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Efficient numerical methods in electronic circuit simulation

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Efficient numerical methods in electronic circuit simulation

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Mathematisches Institut

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Preface

Progress in today's high-technology industries is strongly associated with the development of new mathematical tools. A typical illustration of this partnership is the mathematical modeling and numerical simulation of electric circuits and semiconductor devices.

Recent results in the field of circuit simulation are presented in this report. They were achieved within the "Bayerischer Forschungsverbund für Technisch-Wissenschaftliches Hochleistungsrechnen (FORTWIHR)" and the BMFT project "Anwendungsorientierte Verbundprojekte auf dem Gebiet der Mathematik". They develop new methods in this important and timely field.

Prof. Dr. H. J. Pesch (Technische Universität Clausthal)

Contents

The Simulation of Oscillatory Circuits: An Efficient Integration Scheme

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1 Introduction

Circuit simulation is a standard task for the computer-aided design of electronic circuits. From the mathematical point of view, the transient simulation is well-understood for most circuits. For very large circuits (e. g., memory chips) and highly oscillatory circuits (e. g., quartz oscillators) standard techniques will fail due to the very large computational effort for the numerical simulation. The exploitation of latency is a remedy for very large circuits, see, e. g., [4]. The simulation of highly oscillatory circuits, however, requires a quite different approach.

In this paper, a new discretization scheme will be presented that is able to use some information about the ordinary differential equations (ODEs) arising from the modeling of the electronic circuit. In the next section we will describe the formulation of the network equations. In Sec. 3, the numerical integration method will be presented which is able to simulate highly oscillatory circuits efficiently and reliably. This is confirmed by numerical experiments presented in Sec. 4.

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2 Modeling of the circuit

From a mathematical point of view, an electronic circuit consists of the characteristic element equations for the devices (resistor, MOSFET, etc.) and the device parameters. By means of the modified nodal analysis [7], these element equations are combined to form a system of differential-algebraic equations.

In a first step, complicated devices such as MOSFETs have to be modeled by associated discrete circuits. In the following, the MOSFET model due to Shichman and Hodges [8] will be used. This is the so-called level-1 model in the circuit simulator TITAN of Siemens AG [3]. In Fig. 1.1, a cross-section through an n-MOSFET is given. Many of the properties shown here like the pn-junction or overlap capacitances can be found in the associated discrete circuit. The element equations are omitted in this paper, see, e. g., [1].

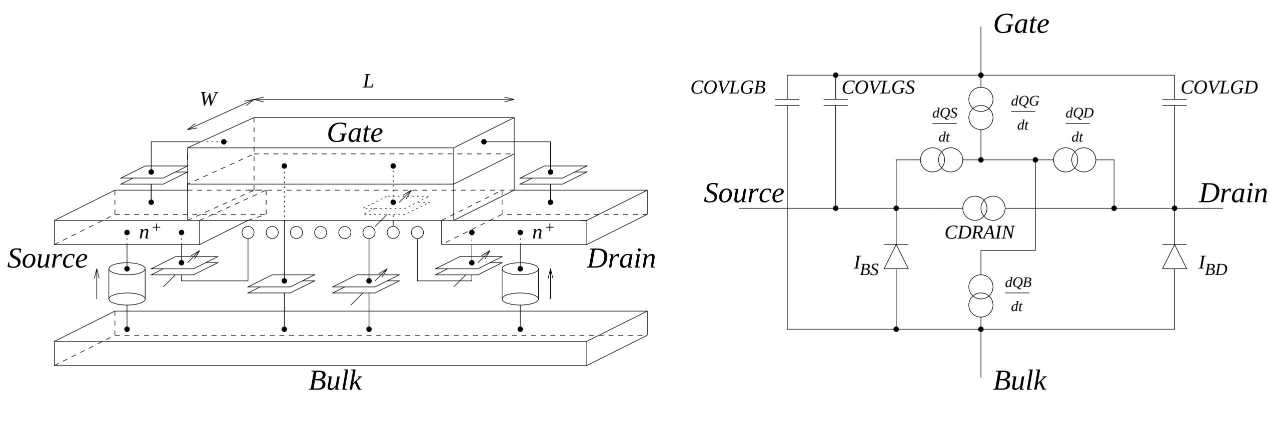

Fig. 1.1 Cross-section through an n-MOSFET device (left) and its associated discrete circuit (right).

The modeling of the circuit results in the network equations, which can be written as

$$C \cdot \dot{x} + B \cdot x = g(t, x), \tag{1.1}$$

where x denotes the vector of nodal voltages, eventually extended by the currents of voltage-controlling elements. C represents the capacitance matrix, B the conductance matrix. As C in Eq. (1.1) is in most cases singular, the transformation of (1.1) into a ODE requires the technique of index reduction, see, e. g., [7]. After the index reduction, the modified matrix in front of $\dot{x}$ is regular and can be inverted. This yields the ODE

$$\dot{x} + A \cdot x = f(x, t), \tag{1.2}$$

where A represents the linear elements of the circuit and $f(x, t)$ assembles the nonlinear parts. Equation (1.2) will be the base for the construction of an efficient integration scheme.

3 EPOCHA: An efficient integration scheme

As standard integration schemes are neither efficient nor reliable for the simulation of highly oscillatory circuits, it is necessary to involve additional information about the circuit into the numerical method. The knowledge of the matrix A in (1.2) is easily available, so this part of the ODE should be taken into account.

The second idea for the construction of an efficient integration scheme is the *Principle of Coherence* which was formulated by Hersch [5] as “Successive approximations should not contradict each other”. The meaning of this principle is best shown with an example. Consider the ODE $\dot{z} + A \cdot z = 0$. The numerical integration for the interval $[t_0, t_1]$ can be carried out with one step with step size $t_1 - t_0$ or two steps with step size $h := (t_1 - t_0)/2$. This yields two approximations:

$$\begin{aligned}
&\text{step size } h\text{:} \quad z(t_1) = \Phi(h) z(t_0 + h) = \Phi^2(h) z(t_0), \\
&\text{step size } 2h\text{:} \ z(t_1) = \Phi(2h) z(t_0).
\end{aligned}$$

According to the Principle of Coherence, these approximations should be equal leading to the condition $\Phi^2(h) = \Phi(2h)$. Therefore, for a coherent integration scheme $\Phi(h) = \exp(-A\, h)$ holds.

This approach for the homogeneous part of (1.2) is combined with a standard multistep approach for the inhomogeneous part. This gives for an explicit method

$$x(t + k\,h) - \exp(-A\,h)\,x(t + (k-1)h) = \\ h\sum_{i=0}^{k} \beta_i\, f(t + (i-1)h, x(t + (i-1)h)),$$

where β_i denotes the coefficients of the inhomongeneous part. In contrast to standard schemes, the coefficients of the method are matrices of the dimension of the ODE. As an example, the coefficients for the explicit scheme with $k = 1$ are given below.

$$\beta_1 = -(I - (I - \exp(-A\,h)) \cdot (A\,h)^{-1}) \cdot (A\,h)^{-1},$$
$$\beta_0 = (I - \exp(-A\,h)) \cdot (A\,h)^{-1} - \beta_1.$$

The analysis of the coherent integration scheme requires some extensions of the well-known theory of standard multistep methods, cf. [2]. It can be shown that the scheme presented here is consistentof order k, it is $A(0)$-stable and therefore convergent. Furthermore, both the explicit and implicit versions of the integration scheme are A-stable without order restrictions. This property seems to be a contradiction to the famous Dahlquist's order barrier. This is not true, as the coefficients of the method depend on the step size h even in the equidistant case in contrast to the methods investigated by Dahlquist.

The coherent integration scheme presented here has been implemented as a fixed step size predictor-corrector scheme with order 1 in the first step and order 2 in the subsequent integration steps. In the following, it will be called EPOCHA for *E*valuating the *P*rinciple *o*f *C*oherence due to *H*ersch with matrix *A*.

4 Simulation of a quartz oscillator

In order to test the integration scheme, a quartz-controlled oscillator for a PLL circuit which is part of an ISDN echo cancel chip was used (courtesy of Siemens AG, Germany). The circuit consists of 4 MOSFETs, 2 capacitances and the quartz, which is modeled by a resistor, an inductance and 2 capacitances. The modeling leads to a 5-dimensional system of type (1.1) of index 1 which is converted by index reduction to a 5-dimensional system of type (1.2). The oscillator was simulated for the interval [0,7 ms] which corresponds to about 250 000 oscillations.

To investigate the efficiency of EPOCHA, the simulation was compared with the results of LSODE-2. This is the well-known integrator LSODE [6] limited to a maximum order of 2 (variable step size, variable order), which is the standard scheme in the circuit simulator TITAN. In order to yield a sufficiently accurate solution it is necessary to chose the prescribed error tolerances for LSODE-2 as 10^{-8}. For the fixed-step-size integrator EPOCHA, the step size has to be set to $h = 2$ ns. In Tab. 1.1, the results show clearly the efficiency of EPOCHA: The gain in CPU time is a factor of about 50. If the maximum order for LSODE is increased to 6, the standard scheme LSODE-6 can use larger step sizes. Nevertheless, the constant step size implementation EPOCHA is more efficient.

	LSODE-2	EPOCHA
integration steps	161 209 213	3 500 000
function evaluations	163 712 516	9 032 499
CPU time (SGI Indigo^2 R4400)	12 171	612

Tab. 1.1 Simulation of the oscillator circuit: Computational effort.

Besides efficiency, reliability is an important property for integration schemes for highly oscillatory ODEs. Varying the prescribed error tolerances and the fixed step size, resp., should be reflected by an only slightly changed solution, the user does not expect totally different characteristics in the solution. In Fig. 1.2, the phase diagrams at $t = 6\,\mathrm{ms}$ are given for various error tolerances and step sizes, resp. For varying step sizes, EPOCHA gives the expected behavior: The accuracy of the solution shows some improvements while the character of the solution remains similar. Varying the prescribed error tolerances for LSODE-6, however, gives three different types of solution: for TOL$= 10^{-4}$ the oscillatory behavior of the circuit has even disappeared!

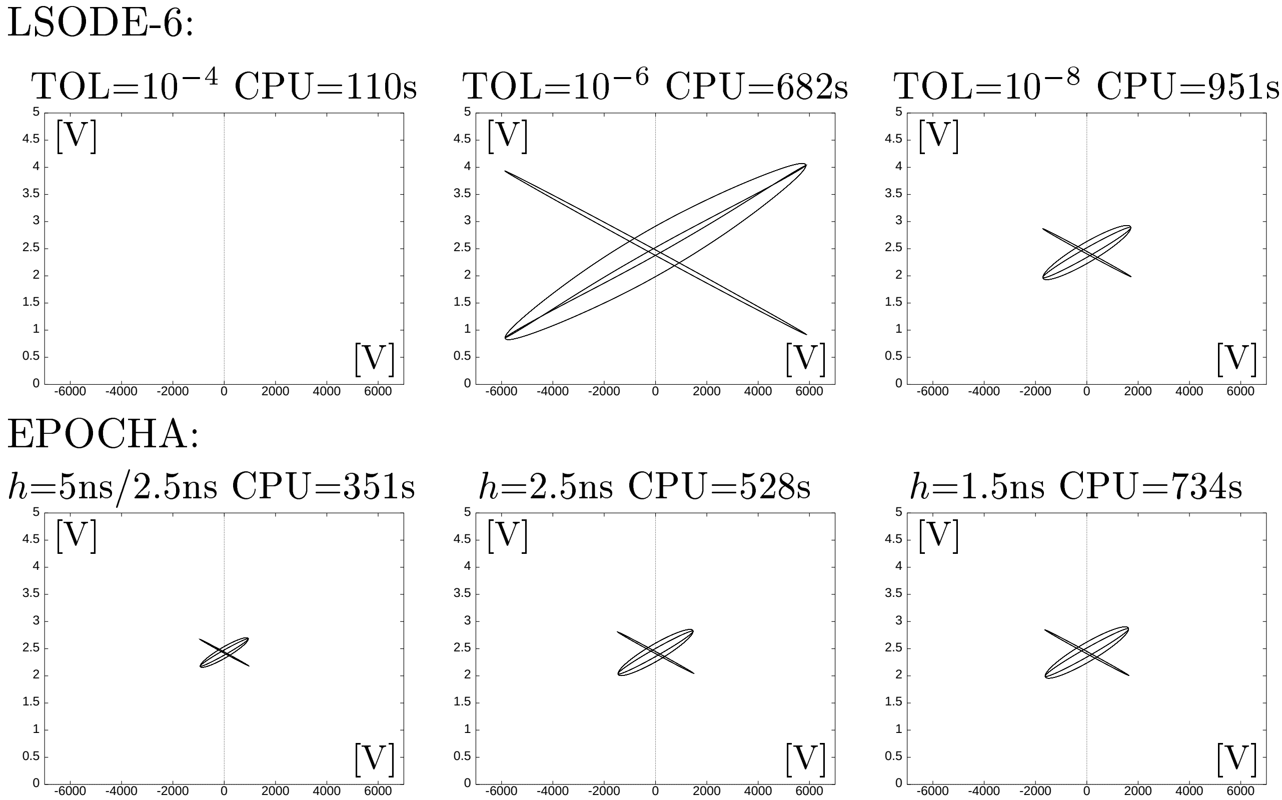

Fig. 1.2 Phase diagrams at $t = 6\,\mathrm{ms}$ for various error tolerances (LSODE-6) and step sizes (EPOCHA). CPU denotes the CPU time necessary for the whole simulation, h = 5ns/2.5ns means h = 5ns for $t \in$ [0,5ms], h = 2.5ns for $t \in$ [5ms,7ms].

Besides the amplitude of an oscillatory solution, its phase should be reflected correctly by the numerical integration scheme. For standard integration schemes it is necessary to use rather small step sizes in order to yield sufficiently accurate results. EPOCHA does not suffer from these problems. In Fig. 1.3, the phase lag at the end of the integration interval is given for one nodal voltage. Though LSODE-6 needs more CPU time, the phase lag is significantly larger.

LSODE-6: TOL=10^{-7}, CPU=681s   EPOCHA: h=2.5ns, CPU=528s

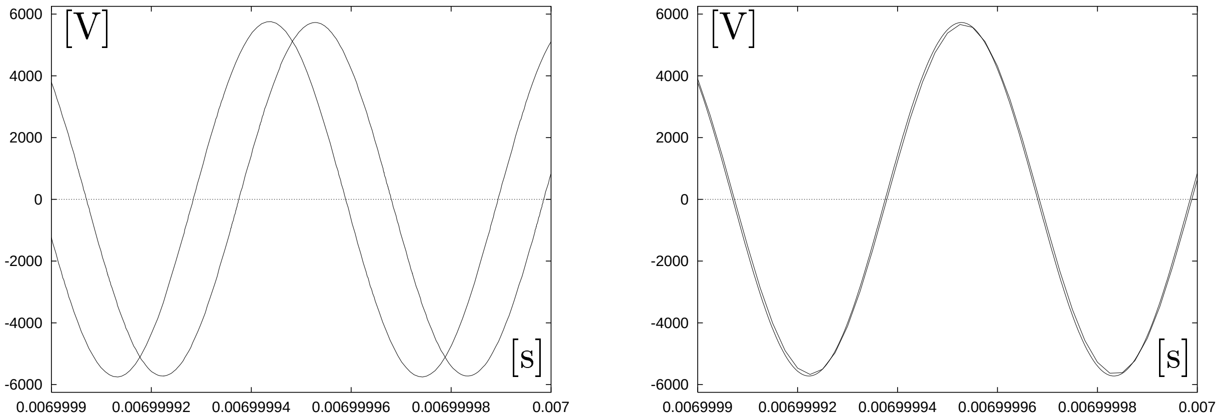

Fig. 1.3 Phase lag at $t = 7.0\,\text{ms}$.

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References

[1] Antognetti, P.; Massobrio, G. (eds.): Semiconductor device modeling with SPICE. McGraw-Hill, 1988.

[2] Denk, G.: A new numerical method for the integration of highly oscillatory second-order ordinary differential equations. APNUM **13** (1993), 57–67.

[3] Feldmann, U.; Wever, U.; Zheng, Q.; Schultz, R.; Wriedt, H.: Algorithms for modern circuit simulation. AEÜ **46** (1992), 274–285.

[4] Günther, M.; Rentrop, P.: Multirate ROW methods and latency of electric circuits. APNUM **13** (1993), 83–102.

[5] Hersch, J.: Contribution à la méthode des équations aux différences. ZAMP **IXa(2)** (1958), 129–180.

[6] Hindmarsh, A. C.: ODEPACK, a systemized collection of ODE solvers. Lawrence Livermore National Laboratory, Report UCRL-88007, 1982.

[7] Kampowsky, W.; Rentrop, P.; Schmidt, W.: Classification and numerical simulation of electric circuits. Surv. Math. Ind. **2** (1992), 23–65.

[8] Shichman, H.; Hodges, D. A.: Insulated-gate field-effect transistor switching circuits. IEEE J. Solid State Circuits **SC-3** (1968), 285–289.

Numerical solution of differential-algebraic equations in electric circuit simulation

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Abstract

The CAD-based automatic generation of electrical network equations leads to differential-algebraic equations. To achieve a both efficient and reliable simulation tool, numerical integration software must be tailored to the special structure and properties of these systems. Hence alternatives to the classical BDF-approach are discussed. Numerical simulation results of a charge pump circuit show the merits of charge-oriented ROW-methods.

1 Introduction

Numerical simulation in the time domain is an important tool in the design of electric circuits nowadays [10]. To be suitable for MOS-circuits, a charge-oriented modeling is applied. Since the network equations of highly integrated circuits with thousands of elements cannot be derived analytically, commercial CAD-based simulation packages are used to generate the equations automatically. This approach, however, involves a redundant set of variables and thus results in differential-algebraic equations (DAEs).

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To solve these systems efficiently, as well as reliably, numerical integration software must be tailored to the special structure and properties of the network equations. After a glance at CAD-based electric network modeling in Sec. 2, we compare the classical BDF-ansatz with new approaches. The numerical simulation results of a charge pump circuit in Sec. 4 show that charge-oriented ROW-type methods are suitable for MOS-circuits.

2 Basic facts on CAD-based electric network modeling

Electronic circuits are modeled by assembling idealized basic linear elements like resistors and capacitors, as well as nonlinear ones like semiconductor devices. The electric behavior of a circuit in the time domain is described by the waveforms of the potentials at each node between two or more adjacent elements and of the branch currents.

Basically three kinds of equations are availabe to derive the network equations. The first two reflect the network topology, whereas the last one describes the type of the elements and their electrical value [7]:

- *Kirchhoff's current law* (KCL): the sum of branch currents leaving each node is zero,
- *Kirchhoff's voltage law* (KVL): the sum of branch voltages along a closed loop is zero, and
- *constitutive element equations* describing the electrical characteristics of basic elements.

In network theory, many different methods for setting up the network equations have been discussed [7]. To be suitable for CAD-based simulation packages, the method must generate the equations *automatically* for *any* circuit with a possibly *small* set of unknowns. The scheme of Modified Nodal Analysis (MNA) [9], widely used in industrial packages such as TITAN [2], meets these demands. In MNA, the following equations are explicitely used:

- KCL for each node and
- the element equations for each voltage defining element, i. e. voltage sources and inductors,

while KVL and the current defining relations of the other elements are inserted directly. Depending on the formulation of constitutive equations for charge-storing elements [1], two different approaches have to be distinguished: the *capacitance-oriented* and the *charge-oriented* one.

The capacitance-oriented approach

In this conventional approach, the constitutive element equation for a capacitor is defined in terms of branch current i and branch voltage u as follows: $i = C(u) \cdot \dot{u}$. The voltage-depending proportionality factor $C(u)$ is the so-called nonlinear differential capacitance.

Hence, applying MNA, the conventional approach yields the capacitance-oriented network equations

$$C(X) \cdot \dot{X} = f(X) + s(t) \tag{2.1}$$

with $X \in \mathbb{R}^n$, $C : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $s : \mathbb{R} \to \mathbb{R}^n$.

X includes all node potentials and currents through voltage-defining elements. The matrix $C(X)$ contains all solution-dependent capacitances and inductances. Resistors yield the linear part of f, and the nonlinear parts are due to nonlinear elements like diodes and transistors. The function s stands for the time dependent current and voltage sources.

[1] The following investigations are similar for circuits containing flux-storing elements like inductors.

If the circuit contains no voltage source and has capacitors at each node to ground, $C(X)$ is regular. In general, the matrix is singular and (2.1) describes a differential-algebraic system in quasilinear-implicit form.

The charge-oriented approach

Today, however, standard VLSI technology is predominantly based on Metal Oxide Silicon Field Effect Transistors (MOSFETs). As usually, nonlinear models composed of basic network elements are used to describe its physical reality to a great extent [7]. However, the intrinsic charge flow in MOSFETs cannot be described with purely reciprocal elements like capacitors [13]. This is due to the fact that charges are nonlinear functions of more than one controlling branch voltage with nonreciprocal derivatives. To manage this problem, the constitutive equations for charge-storing elements are formulated charge-oriented, i. e. relate the charge stored in the element to the terminal voltages [13,15,16].

This approach leads to the charge-oriented network equations in integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(A \cdot q(X)\Big) = f(X) + s(t) \tag{2.2}$$

with $q : \mathbb{R}^n \to \mathbb{R}^m$ and $A \in \mathbb{R}^{n \times m}$. The incidence matrix A assembles the current contributions from the m charges $q(X)$, defined as nonlinear functions of node potentials.

3 Numerical integration of charge-oriented network equations

Since the charge-oriented approach is suitable to model any circuit, most industrial simulation tools like TITAN are based on it. To be workable within the scope of CAD-based circuit simulation, a numerical integration method has to fulfill special demands:

- *Stiffness*
  Owing to the steep voltage courses that correspond to different time scales, electric circuit modeling leads to stiff systems. Hence implicit methods have to be used.

- *Poor smoothness properties*
  The network equations are characterized by a deficiency of smoothness, caused by the imperfect multidimensional fitting of measured parameters in transistor modelling. Methods of lower order up to three have shown to be suitable in this case.

- *Low accuracy requirements*
  The user is only interested in rather qualitative voltage courses. Provided that error tolerances are met reliably, an accuracy of approximately 1% is enough. Hence methods are preferred that are especially efficient at lower accuracy demands.

- *Compatibility with simulation packages*
  Since the integration software is to be built in the simulation environment, it must be compatible with the given data structures. No inaccessible information may be used. For example, the second derivatives of q are not available. Furthermore, often this information does not even exist due to the poor smoothness properties.

Classical approach

Until now, most packages use an unspecific method for arbitrary nonlinear differential-algebraic equations with index ≤ 1 to solve these equations numerically: Gear's backward differentiation formulas [3]. At a given time point t_n, $\frac{\mathrm{d}}{\mathrm{d}t}\, q(X(t_n))$ is approximated in TITAN [2] by $\rho\, q(X_n)/h$, with the difference operator ρ given by the BDF-scheme of order k (in general, $k \in \{1, 2, 3\}$)

$$\rho\, q(X_n) = \alpha \cdot q(X_n) + \sum_{i=1}^{k} \alpha_i \cdot q(X_{n-i}).$$

By substituting this approximation into (2.2) a system of nonlinear equations is obtained in the unknown X_n, which is solved by Newton-type iterations. However, this approach suffers from the short-coming: As charges are used for integration, the user has no direct control on the really interesting circuit variables, i. e. the node potentials X. Time step and error control are based on charges. Alternatives within the BDF-approach are discussed in Denk [1] and Sieber et al. [14].

Alternative approaches

Alternatively, the integration can be based on node potentials by reformulating system (2.2). We distinguish two approaches:

1. *The general approach*

By performing explicitly the differentiation in (2.2), one gets the quasilinear-implicit system

$$A \cdot \frac{\partial q(X)}{\partial X} \cdot \dot{X} = f(X) + s(t)\,. \tag{2.3}$$

The definition

$$C(X) := A \cdot \frac{\partial q(X)}{\partial X}$$

yields a system formally equivalent to the conventional approach (2.1). However, now the "capacitance"-matrix is unsymmetrical due to the nonreciprocal MOS-capacitances.

2. *The integral form approach*

Using the integral form structure in (2.2), it is natural to introduce charges, the quantities of choice in semiconductor physics, as additional state variables. Now (2.2) is embedded into the differential-algebraic system

$$A \cdot \dot{Y} = f(X) + s(t) \tag{2.4a}$$

$$0 = Y - q(X) \tag{2.4b}$$

of linear implicit form.

Comparing both approaches, the latter one seems to be advantageous in the scope of simulation packages: Topology information stored in the incidence matrix is available. Furthermore, no second derivatives of q are necessary — in contradiction to an implicit method applied to (2.3). Hence it is advantageous to construct numerical methods directly based on the linear-implicit network equations (2.4).

Charge-oriented ROW-methods

Exploiting the special structure in (2.4b), we have derived CHORAL (*Ch*arge-*or*iented *al*gorithm), an embedded ROW-type method of order (2)3 with four stages. It is tailored to index-1-systems and special systems of index 2 that offen appear in electric circuit simulation.[2]

Starting with initial values X_0 at t_0, one step with step size h yields the numerical approximation X_1 for X at time point $t_0 + h$

$$X_1 = X_0 + \sum_{i=1}^{\nu} d_i \cdot \kappa_i$$

with the increments κ_i defined by the linear systems

$$\mathcal{M} \cdot \kappa_i = A \cdot (q(X_0) - q(a_i)) + h \cdot \sum_{j=1}^{i} \beta_{ij} \cdot \{f(a_j) + s(t_0 + h\alpha_j)\} +$$
$$h \cdot \sum_{j=1}^{i-1} \beta_{ij} \frac{\partial f}{\partial X}(X_0) \cdot \kappa_j + h^2 \tau_i \cdot \dot{s}(t_0) \quad \text{for}\, i = 1, \ldots, \nu\,.$$

[2]Numerical methods can only be efficient, as well as reliable, if they are tailored to systems of special structure and index. Consequently, the effects of modeling and network structure on these properties are of great interest. Corresponding investigations are made in [7].

We have used the abbreviations

$$\mathcal{M} = \frac{\partial}{\partial X}\left(A \cdot q(X) - h\gamma \cdot f(X) \right)\bigg|_{X=X_0}$$

and

$$a_i = X_0 + \sum_{j=1}^{i-1} \sigma_{ij} \kappa_j \, .$$

The coefficient set of CHORAL, as well as details of construction and implementation, are discussed in [6]. A special coefficient set for index-1-systems is given in [5].

CHORAL fulfills the special requirements mentioned above. Especially, it is compatible with simulation tools like TITAN: Only evaluations of f, q, s and the Jacobians $\partial f / \partial X$ and $\partial q / \partial X$ are required at discrete points. Furthermore, the matrix $\mathcal{M}$ is just the iteration matrix of the BDF-ansatz. Accordingly, decomposition algorithms for $\mathcal{M}$ in TITAN can be used, provided that they are accurate enough. Furthermore, software developed to evaluate currents and charge contributions element by element can be used by CHORAL, too.

Summing up, CHORAL is an alternative to current integration methods for use in simulation packages. In contradiction to the widely used BDF-approach, the error is controlled in the state variables that are of interest to the user. A time step prediction based on these quantities is offered.

4 Numerical example: Charge pump circuit

Charge-oriented ROW-methods have been tested on several circuits from industrial applications, e. g. logical subcircuits like NOR-gate and NAND-gate [5,8] and more complex digital circuits like two-bit adders [6]. The simulation results of an 8-bit counter obtained by a charge-oriented extrapolation method are discussed in [4].

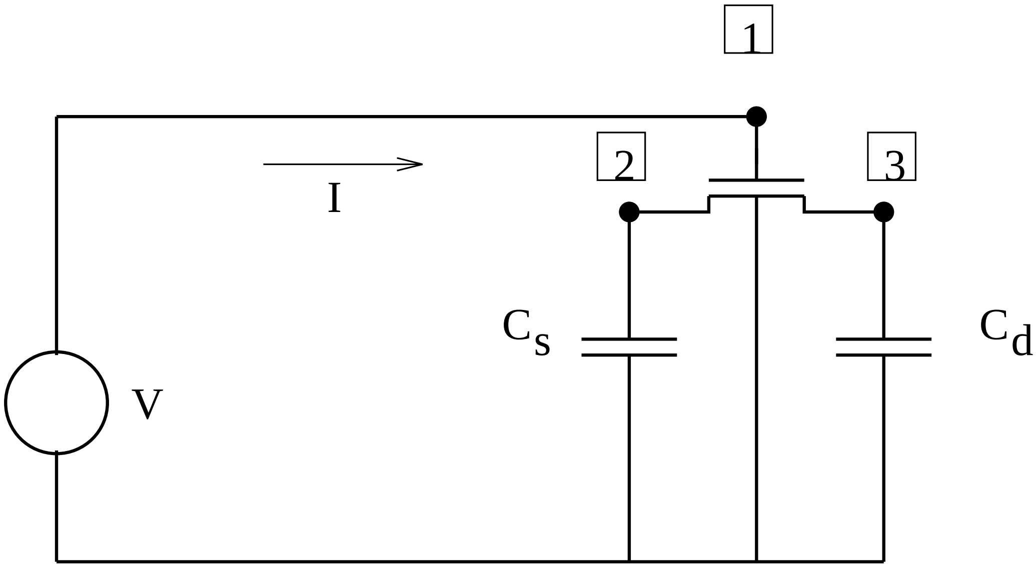

Fig. 2.1 Charge pump circuit consisting of a MOS-transistor and two linear capacitances

The charge pump circuit [13] shown in Fig. 2.1 serves as an instructive test example for the suitability of numerical methods applied to MOS-circuits. By neglecting the current flow in the channel and all overlap and p-n junction capacitances, the movement of charges due to the intrinsic capacitances in MOS-transistors is highlighted. Consequently, the circuit is turned into a nonlinear capacitance divider: With every cycle of the input signal V_{in}, charge is pumped into and out of the MOS-transistor, as well as to the external linear capacitances C_s and C_d. Hence the source and drain voltages oscillate similarly. The amplitude, however, depends on the ratio of the capacitances.

We have compared the conventional Meyer-model [12] that yields capacitance-oriented network equations and its charge-oriented extension [6]. The results for both models are shown in Fig. 2.2. In the conventional model, the amplitude of the oscillation shows a rather random behavior — in contradiction to charge-conservation. Such irregularities are caused by the inappropriate modeling, and cannot be prevented by imposing higher accuracy demands. On the contrary, the simulation of the charge-model by CHORAL yields physically correct results even at low accuracy requirements.

Concluding, we have: Only charge-oriented models are suitable to

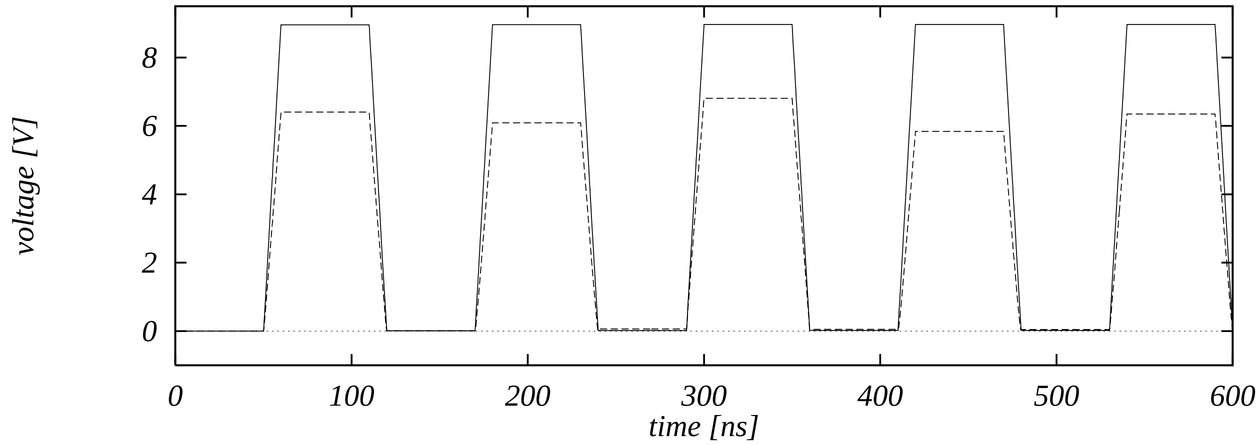

Fig. 2.2 Charge pump circuit: Voltage course of source node at low error tolerance The results for the capacitance-oriented model (- - -) was obtained by ROWDAIND2 [11], and for the charge-oriented extension (—) by CHORAL.

describe the intrinisic charge flow in MOS-transistors correctly. Hence numerical integration must be based on the charge-oriented approach. For this purpose, CHORAL has shown to be suitable.

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References

[1] Denk, G.: An improved numerical integration method in the circuit simulator SPICE2-S. In Bank, R. E. et al. (ed.): Mathematical modelling and simulation of electrical circuits and semiconductor devices. ISNM 93, Birkhäuser Verlag, Basel (1990), pp. 85 – 99.

[2] Feldmann, U.; Wever, U. A.; Zheng, Q.; Schultz, R.: Algorithms for modern circuit simulation. Archiv für Elektronik und Übertragungstechnik **46** (1992), pp. 274 – 285.

[3] Gear, C. W.: Simultaneous numerical solution of differential-algebraic equations. IEEE Trans. Circuit Theory **CT–18** (1971), pp. 89 – 95.

[4] Gerstberger, R.; Günther, M.: Charge-oriented extrapolation methods in digital circuit simulation. To appear in Appl. Numer. Math.

[5] Günther, M.: Charge-oriented modelling of electric circuits and Rosenbrock-Wanner methods. To appear in Journal of Computing and Information.

[6] Günther, M.: Ladungsorientierte Rosenbrock-Wanner-Methoden zur numerischen Simulation digitaler Schaltungen. Dissertation, Technische Universität München, Institut für Informatik (1995).

[7] Günther, M.; Feldmann, U.: The DAE-index in electric circuit simulation. To appear in Mathematics and Computers in Simulation.

[8] Günther, M.; Rentrop, P.: Suitable one-step methods for quasilinear-implicit ode's. Report TUM-M9405, Mathematisches Institut, Technische Universität München (1994).

[9] Ho, C.-W.; Ruehli, A. E.; Brennan, P. A.: The modified nodal approach to network analysis. IEEE Trans. Circuits and Systems **CAS-22** (1975), pp. 505 — 509.

[10] Kampowsky, W.; Rentrop, P.; Schmidt, W.: Classification and numerical simulation of electric circuits. Surv. Math. Ind. **2** (1992), pp. 23 – 65.

[11] Lubich, Ch.; Roche, M.: Rosenbrock Methods for differential-algebraic systems with solution-dependent singular matrix multiplying the derivative. Computing **43** (1990), pp. 325 – 342.

[12] Meyer, J. E.: MOS models and circuit simulation. RCA Rev. **32** (1971), pp. 42 – 63.

[13] Sakallah, K. A.; Yen, Y.; Greenberg, S. S.: A first-order charge conserving MOS capacitance model. IEEE Trans. Computer-Aided Design **CAD-9** (1990), pp. 99 – 108.

[14] Sieber, E.-R.; Feldmann, U.; Schultz, R.; Wriedt, H.: Timestep control for charge conserving integration in circuit simulation. In Bank, R. E. et al. (ed.): Mathematical modelling and simulation of electrical circuits and semiconductor devices. ISNM 117, Birkhäuser Verlag, Basel (1994), pp. 103 – 113.

[15] Ward, D. E.; Dutton, R. W.: A charge-oriented model for MOS transistor capacitances. IEEE J. Solid-State Circuits **SC-13** (1978), pp. 703 – 708.

[16] Yang, P.; Epler, B. D.; Chatterjee, P. K.: An investigation of the charge conservation problem for MOSFET circuit simulation. IEEE J. Solid State Circuits **SC-18** (1983), pp. 128 –138.

A Glance at the Stability of Integrated Circuits with Oscillating Behaviour

Petra A. Selting*, H.-J. Bauer *, W. Schmidt **

1 Oscillating Electric Circuits

In the last decade design methods for electric circuits have developed enormously. Today circuit simulation programmes are standard tools for computer aided development. A special task in this field is the investigation of oscillating circuits, more precisely, the analysis and simulation of oscillating electric circuits with constant input.

The ring oscillator (fig. 1) is a typical example of the above mentioned class. It consists of a series of coupled MOS-FET-inverters. The output signal of the last inverter is reinjected to the first one. Zheng and Dellnitz (1990) discuss this example from both the analytical and numerical point of view. Kampowsky et. al. (1992) investigate in the computation of limit cycles. Schmidt (1993, 1994) replaces their simple MOS-FET-model by the model of Shichman and Hodges and compares MOS-FET-models of various simulation levels.

The second example, the Clapp oscillator (fig. 2), is based on a npn-transistor. It changes a constant input signal to an almost sinusoidal output signal (fig. 3). Because of this input-output behaviour, this circuit is of practical interest, especially in high frequency applications. The model equations of this circuit cover several pages and are dropped here. A complete description, including all parameters, is given by Kärtner (1992).

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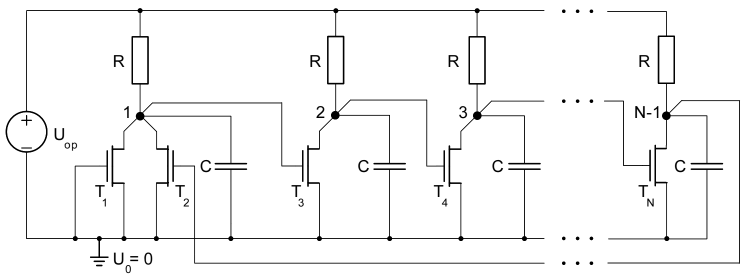

Figure 1: Ring Oscillator

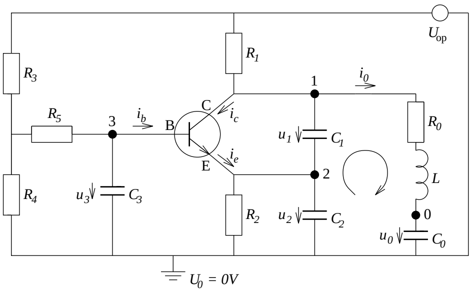

Figure 2: Clapp Oscillator

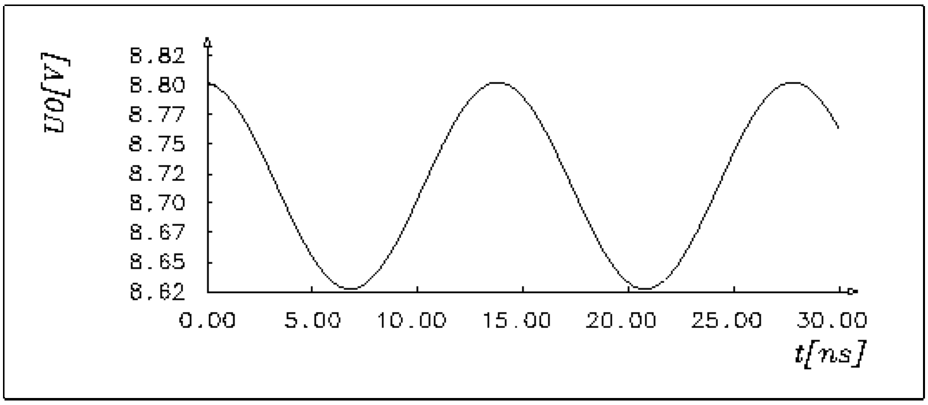

Figure 3: Output Signal of the Clapp Oscillator

2 Computation and Stability of Limit Cycles

The basic tools to translate a circuit diagram, as shown in fig. 1 or fig. 2, into the formalism of mathematics are Kirchhoff's laws:

- The sum of currents traversing each cutset of the network equals zero.
- The sum of voltages around a closed loop of the network equals zero.

With these tools the time-dependent behaviour of an electric circuit with constant input can often be described by an autonomous system of implicit differential equations

$$f(y(t), \dot{y}(t)) = 0, \ \ y(0) = y_0, \quad y_0 \in \mathbb{R}^n, \ \ f : \mathbb{R}^{2n} \to \mathbb{R}^n, \tag{3.1}$$

where $f_{\dot{y}}$ is supposed to be regular. The highly nonlinear function f represents the topology of the network and the model functions of the electric elements, like transistors, diodes etc. Usually the model functions are fitted data of electronic measurements, therefore we cannot expect f to be very often differentiable. Compared to general circuit simulation, the dimension n of the system is rather low, typically n is in the range of 5 to 1000.

To compute a periodic solution of (3.1), we must determine the unknown period $T > 0$ and a set of initial values $y \in \mathbb{R}^n$, such that the relation $y(t+T) = y(t)$ holds for all $t \geq 0$. In the case of ordinary differential equations Seydel (1982) proposes to solve a $(n+1)$-dimensional boundary value problem for this task. In the case of autonomous differential equations we get

$$\begin{pmatrix} f(y, \dot{y}) \\ \dot{T} \end{pmatrix} = 0, \qquad y(0) = y(T), \tag{3.2}$$

To obtain uniqueness for this $(n+1)$-dimensional system, the phase shift can be fixed by an additional boundary condition. For example, we can prescribe an initial value for an arbitrary component y_k by

$$y_k(0) - \eta_k = 0, \qquad \eta_k \in \mathbb{R}, \quad 1 \leq k \leq n. \tag{3.3}$$

Kampowsky et. al. (1992) solve the enlarged boundary value problem (3.2), (3.3) with the multiple shooting method. Schmidt (1994) gives some hints to adapt this method to limit cycle computation. Unfortunately the multiple shooting method depends on sufficiently accurate starting values, otherwise the convergence cannot be guaranteed.

In general, computation of starting values is a rather time consuming task, which is often based on the principle of trial and error. In the case of limit cycle computation, this can be done automatically by switching between the solution of a boundary value problem and an initial value problem. In order to obtain sufficiently accurate starting values, we solve an initial value problem with arbitrary initial values. During the integration process, we look for some periodic behaviour of the solution components y_k, $1 \le k \le n$, and estimate the period T and an initial vector $y(0)$. With this estimation the solution of the boundary value problem (3.2), (3.3) is started. If the multiple shooting method does not converge fast enough, the iteration is stopped and the initial value problem is integrated again with the initial values determined by multiple shooting, and so on. Moreover, the step size control of the integrator of the initial value problem gives information about the choice of the shooting intervals. In other words, the number and distribution of the shooting nodes are determined adaptively.

In a production process the parameters of the electronic devices will vary in between technically caused tolerances. Therefore, the next step after the computation of a periodic solution is to answer, whether the solution is stable or not. Let $\varphi(t, y_o)$ be a periodic solution with period T of the initial value problem

$$f(y, \dot{y}) = 0, \qquad y(0) = y_0. \tag{3.4}$$

We investigate the sensitivity of φ against perturbations of the initial vector y_0, i. e. we replace y_0 by $\hat{y} := y_0 + \varepsilon$, $\varepsilon \in \mathbb{R}^n$. Furthermore $f_{\dot{y}}$ is assumed to be regular in an environment of φ. Taylor expansion of the difference $d(t)$ between exact and perturbed solution at $t = T$ yields:

$$d(T) := \varphi(T, y_0 + \varepsilon) - \varphi(T, y_0) = \frac{\partial \varphi(T, y_0)}{\partial y_0} \varepsilon + \mathcal{O}(\varepsilon^2). \tag{3.5}$$

The *monodromy matrix*

$$M := \frac{\partial \varphi(T, y_0)}{\partial y_0} \tag{3.6}$$

tells us, whether the initial pertubation ε decays or increases. However, φ is the result of a numerical process and is not available in explicit terms. Some properties of φ help us to compute the monodromy matrix M numerically.

The periodic solution φ satisfies the differential equation

$$f(\varphi(t, y_0),\, \dot{\varphi}(t, y_0)) = 0. \tag{3.7}$$

Differentiation with respect to y_0 yields

$$\frac{\partial f}{\partial y}\frac{\partial \varphi(t, y_0)}{\partial y_0} + \frac{\partial f}{\partial \dot{y}}\frac{d}{dt}\frac{\partial \varphi(t, y_0)}{\partial y_0} = 0. \tag{3.8}$$

Thus

$$M = \Phi(T), \qquad \text{where} \qquad \Phi(t) := \frac{\partial \varphi(t, y_0)}{\partial y_0} \tag{3.9}$$

and Φ is a solution of the linear differential equation

$$f_{\dot{y}}(\varphi, \dot{\varphi})\dot{\Phi}(t) + f_y(\varphi, \dot{\varphi})\Phi(t) = 0, \qquad \Phi(0) = I. \tag{3.10}$$

The enormous storage requirement restricts the computation of M by (3.10) to small systems. As a remedy the monodromy matrix can be calculated columnwise.

3 Characterization of Stability by Eigenvalues

The stability of a periodic solution can be determined by Floquet-theory and the eigenvalues of the monodromy matrix. Let us assume $\varphi(t, y_0)$ to be a periodic solution of (3.4) and the corresponding monodromy matrix M to have the n eigenvalues $\mu_1, \ldots, \mu_n$ with $|\mu_n| = 1$. Then φ is *stable* (with respect to pertubations of the initial values y_0), if $|\mu_j| < 1$ for $j = 1, \ldots, n-1$, otherwise φ is *unstable*.

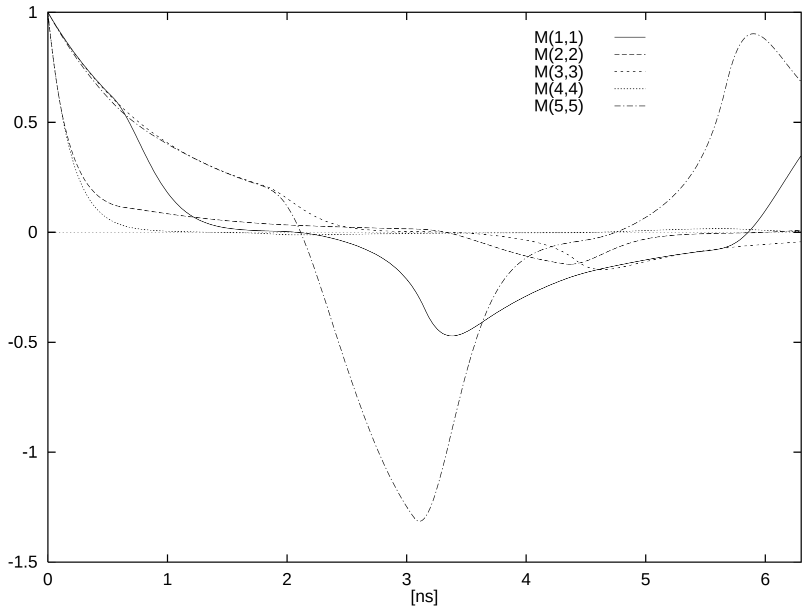

Figure 4: Diagonal Elements of the Monodromy Matrix

As a numerical example let us consider the ring oscillator of fig. 1 with five inverters. A periodic solution was determined by multiple shooting. Figure 4 shows the diagonal elements of the solution of the initial value problem for the monodromy matrix. Their complex eigenvalues have been computed with a standard QR-algorithm

$$
\begin{aligned}
&(-3.684317 - 9.2888468i) \cdot 10^{-7}, \\
&(-3.684317 + 9.2888468i) \cdot 10^{-7}, \\
&-1.339028 \cdot 10^{-4}, \quad 3.894091 \cdot 10^{-4}, \quad 0.999989.
\end{aligned}
$$

Four of these eigenvalues are inside the unit circle, the fifth is on its boundary, thus the periodic orbit is stable, as expected.

4 Implementation in TITAN

The outlined algorithm is implemented in the circuit simulation programme TITAN, devoloped by Siemens AG, Munich. In practice some additional difficulties arise. First, the equations of a circuit are not given in symbolic form, but TITAN evaluates the function values and the Jacobian at discrete time steps. Second, the basic algorithms and equations are formulated in the charge oriented approach

$$A\dot{q}(y(t)) + f(y(t)) = 0, \qquad y(0) = y_0, \tag{3.11}$$

where $y \in \mathbb{R}^n$ is the vector of node potentials and currents through voltage defining elements and $q \in \mathbb{R}^m$ is the vector of charges. With some straight forward calculations equation (3.11) can be handled in a quite similar way as (3.7) - (3.10). This charge oriented approach applied to the ring oscillator yields equivalent results as the computer simulation in the previous section.

References

[1] Kärtner, F.X.: Untersuchung des Rauschverhaltens von Oszillatoren, Dissertation, Technische Universität München (1989).

[2] Kampowsky W.; Rentrop P.; Schmidt W.: Classification and numerical simulation of electronic circuits, Surv. Math. Ind. 2, (1992), pp. 23-65.

[3] Schmidt W.: Computation of Periodic Orbits for Oscillating Electric Circuits – Comparison of Various MOS-FET Models, U. Helmke, R. Mennicken, J. Saurer (eds): Systems and Networks: Mathematical Theory and Applications, Vol. II (1993), pp. 467 - 470.

[4] Schmidt W.: Limit Cycle Computation of Oscillating Electric Circuits, International Series of Numerical Mathematics, Vol. 117, Basel (1994), pp. 91-101.

[5] Seydel R.: Numerical computation of periodic orbits that bifurcate from stationary solutions of ODEs, Appl. Math. Comp. 9 (1981), pp. 257-271.

[6] Zheng, Q.; Dellnitz, M.: Schwingungen eines Ringoszillators - eine numerische Behandlung unter Berücksichtigung der Symmetrie, ZAMM, Z. angew. Mech. 70, (1990) 4, T135 - T138.

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Optimum Design of Microwave Oscillators with Minimized Phase Noise

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1 Introduction

A low phase noise is besides the signal properties essential to the design of oscillating electrical circuits. Up to now, a reduction of the phase noise can only be achieved by using empirical rules which, e. g., require additional elements in the circuit or the manufacture of lines of prototypes (trial and error).

A new and general method to minimize the single-sideband phase noise of free running oscillators is presented. It is based on the description of the signal and noise behavior of an oscillator circuit by the Langevin equations

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$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \boldsymbol{\xi}(t), y(t)) \\
&= \mathbf{f}(\mathbf{x}(t)) + \mathbf{G}(\mathbf{x}(t)) \cdot \boldsymbol{\xi}(t) + \mathbf{g}(\mathbf{x}(t)) \cdot y(t) + \mathcal{O}(\boldsymbol{\xi}^2, y^2, \boldsymbol{\xi} y), \\
\mathbf{G}(\mathbf{x}(t)) &= \frac{\partial \mathbf{f}(\mathbf{x}(t), \boldsymbol{\xi}(t), y(t))}{\partial \boldsymbol{\xi}} |_{\boldsymbol{\xi}=0, y=0}, \qquad (4.1) \\
\mathbf{g}(\mathbf{x}(t)) &= \frac{\partial \mathbf{f}(\mathbf{x}(t), \boldsymbol{\xi}(t), y(t))}{\partial y} |_{\boldsymbol{\xi}=0, y=0},
\end{aligned}
$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$ are the state variables of the circuit (voltages, currencies), $\boldsymbol{\xi}$ are the white noise sources and y is a nonlinear $f^{-\alpha}$ noise source denoting the baseband noise. The noise sources are assumed to be small compared to the signal amplitudes and the terms of order $\mathcal{O}(\boldsymbol{\xi}^2, y^2, \boldsymbol{\xi} y)$ are neglected.

2 Single-Sideband Phase Noise

The single-sideband phase noise $L(f_m)$ of free running oscillators can be calculated by solving Eq. (4.1) and using a perturbation approach (Kärtner [5])

$$
L(f_m) = \frac{\Delta f_{3dB}}{\pi f_m^2} + \omega_0^2 |g_{1,0}|^2 \frac{c}{|2\pi f_m|^{2+\alpha}} \qquad (4.2)
$$

$$
\text{with} \quad \Delta f_{3dB} = \frac{1}{4\pi} \omega_0^2 \frac{1}{T_0} \int_0^{T_0} \mathbf{v}(\mathbf{x}(t))^T \mathbf{G}(\mathbf{x}(t)) \boldsymbol{\Gamma}(\mathbf{x}(t)) \mathbf{G}(\mathbf{x}(t))^T \mathbf{v}(\mathbf{x}(t)) \mathrm{d}t
$$

$$
\text{and} \quad g_{1,0} = \frac{1}{T_0} \int_0^{T_0} \mathbf{v}(\mathbf{x})^T \mathbf{g}(\mathbf{x}) \mathrm{d}t.
$$

The first term on the right hand side of Eq. (4.2) describes the phase noise caused by the white noise sources. $\mathbf{v}(\mathbf{x}(t))$ is the left-sided eigenvector of the eigenvalue 1 of the fundamental matrix $\boldsymbol{\Psi}(T_0, 0)$. The matrix $\boldsymbol{\Gamma}$ denotes the correlation matrix of the white noise sources. The second term of Eq. (4.2) describes the phase noise caused by the baseband noise. The factor $g_{1,0}$ characterizes the upconversion of the baseband noise to the carrier frequency $\omega_0 = 2\pi/T_0$. The modulation of the $f^{-\alpha}$ noise source due to the oscillation is taken into account as well as the upconversion of the baseband noise caused by the nonlinearities in the circuit. The factor c is derived from baseband noise measurements. f_m is the offset frequency where the single-sideband phase noise is to be calculated. The functions $\mathbf{x}(t)$ and $\mathbf{v}(t)$ depend on the design parameters $\mathbf{p}$ of the circuit (e. g., capacitors and inductors of the linear network) by the system of $2n$ nonlinear differential equations derived from (4.1)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}), \tag{4.3}$$

$$\dot{\mathbf{v}}(t) = -\left[\frac{\partial \mathbf{f}(\mathbf{x}(t), \mathbf{p})}{\partial \mathbf{x}}\right]^T \cdot \mathbf{v}(t), \quad t \in [0, T_0] \tag{4.4}$$

and $2n+1$ boundary conditions

$$r(x(0), v(0), x(T_0), v(T_0)) \;=\; 0. \tag{4.5}$$

3 Optimal Control Problem

The minimization of the phase noise $L(f_m)$ with respect to the design parameters $\mathbf{p}$ and subject to Eqs. (4.3) – (4.5) is an optimal control problem with $\mathbf{p}$ as (constant) control function from a finite dimensional control space and $\mathbf{x}$ and $\mathbf{v}$ as state variables of the optimal control problem. The optimization problem is solved numerically by the direct collocation method DIRCOL [9]. The method is based on a discretization of the state variables by piecewise cubic spline functions satisfying the dynamic equations at the grid points of a time grid and at the centers between (cf. Hargraves, Paris [4] and [8]). By this approach, the optimal control problem is transformed to a (finite dimensional) nonlinearly constrained optimization problem that can be solved by the Sequential Quadratic Programming method due to Gill et al. [2]. Compared with other methods for solving optimal control problems the direct collocation method is easy-to-use (as knowledge of optimal control theory is not required), robust (not much information on the solution is a priori required) and reliable (if accuracy requirements are not too high) [9].

4 Numerical and Experimental Results

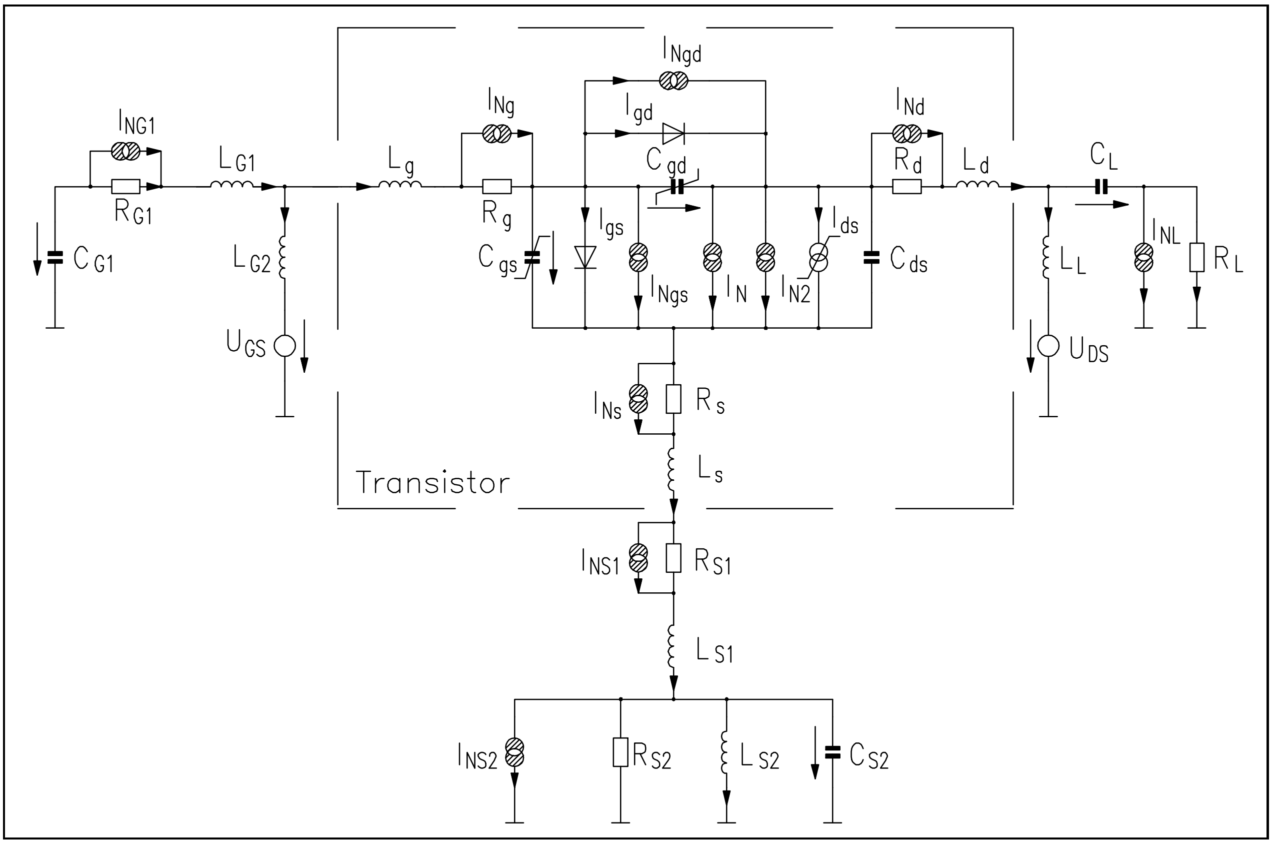

Fig. 1 Equivalent circuit of the oscillator to be optimized.

The new method to improve the design of oscillators is applied to minimize the single-sideband phase noise of a planar integrated [3] free running microwave oscillator at 15 GHz [1], [6]. A GaAs MESFET NE710 is used for the active element which is connected at the gate and source terminals with microstrip lines as resonators. The equivalent circuit of the oscillator is depicted in Fig. 1. In this special case, the five design parameters L_{G1}, C_{G1}, L_{S1}, L_{S2}, and C_{S2} of the linear network are optimized subject to a system of 20 highly nonlinear differential equations (4.3), (4.4). Only the upconverted baseband noise to the carrier frequency is taken into account to minimize the single-sideband phase noise $L(f_m)$ in Eq. (4.2). After the computation of the optimal values of the five design parameters a prototype MIN2 of the new designed oscillator has been manufactured [6] (Fig. 2).

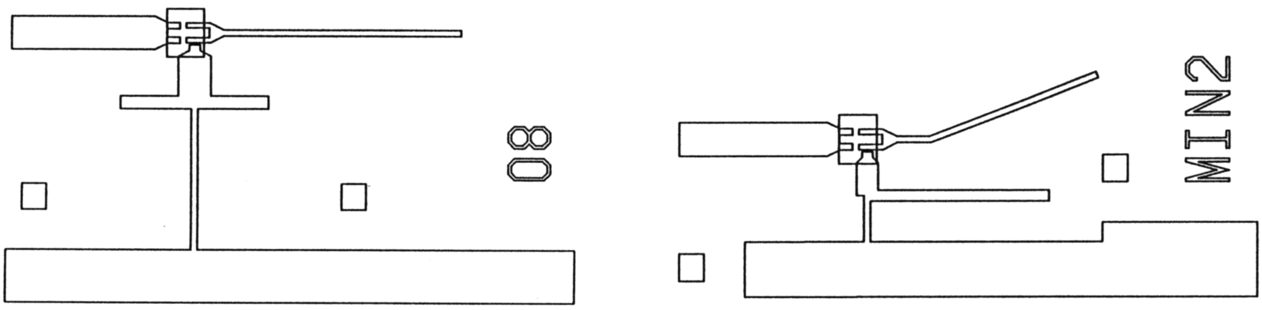

Fig. 2 Layout of the original oscillator O8 (left side) and of the optimized oscillator MIN2 (right side) with a scaling factor of about 10:1.

The measured output power of the reference oscillator O8 is 12.7 dBm compared with the output power of 14.4 dBm of the oscillator with the minimized phase noise. For the prototype MIN2 a reduction of 8 dB of the phase noise caused by the upconverted baseband noise is measured at a frequency deviation of 10 kHz compared to the original oscillator O8 (Fig. 3) [1], [6]. The difference of the measured (——)

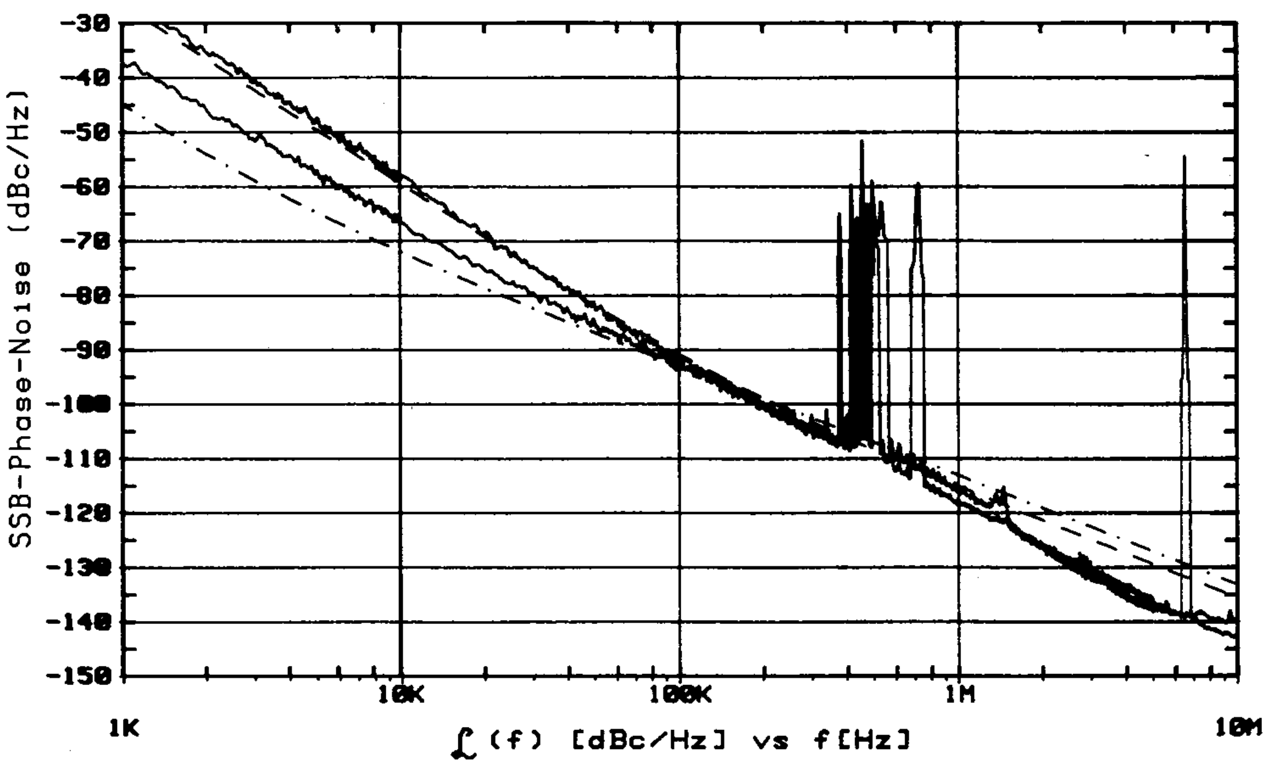

Fig. 3 Measured (——) and simulated single-sideband phase noise of the original oscillator (– – –) and the oscillator with minimized phase noise (–·–·–).

from the calculated single-sideband phase noise (– · – · –) of the optimized oscillator is mainly due to deviations in the technical data of the elements used to manufacture the prototype MIN2. The measured peaks between 400 kHz and 7 MHz are due to external interferences.

5 Conclusions

A new method for the design of microwave oscillators with minimized phase noise has been outlined and successfully applied to reduce the single-sideband phase noise of a certain high frequent oscillator. With the new method a significant reduction of the phase noise is achieved in both simulation and experiment. Hereby neither the use of additional elements in the circuit nor the manufacture of lines of prototypes in advance is required.

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References

[1] Anzill, W.; von Stryk, O.; Bulirsch, R.; Russer, P.: Phase Noise Minimization of Microwave Oscillators by Optimal Design. IEEE MTT-S Symposium Digest, Orlando, Florida, USA, 1995.

[2] Gill, P. E.; Murray, W.; Saunders, M. A.; Wright, M. H.: User's guide for NPSOL (Version 4.0). Report SOL 86-2, Dep. of Operations Research, Stanford University, California, USA (1986).

[3] Güngerich, V.; Schadel, R.; Ramisch, R.; Russer, P.: A process for inserting chips into planar microwave structures on semiconductor substrates. Journal Microelectronic Engineering **18** (1992) 247-252.

[4] Hargraves, C. R.; Paris, S. W.: Direct trajectory optimization using nonlinear programming and collocation. AIAA J. Guidance **10**, 4 (1987) 338-342.

[5] Kärtner, F.: Analysis of white and $f^{-\alpha}$ noise in oscillators. Int. Journal of Circuit Theory and Applications **18** (1990) 485-519.

[6] Schöbel, R.: Untersuchung und Optimierung des Phasenrauschens eines Mikrowellenoszillators in Abhängigkeit vom linearen Schaltungsteil. Diploma thesis, Lehrstuhl f. Hochfrequenztechnik, Technische Universität München (1993).

[7] Stoer, J.; Bulirsch, R.: Introduction to Numerical Analysis. 2nd ed. (Springer, 1993).

[8] Stryk, O. von: Numerical solution of optimal control problems by direct collocation. In: R. Bulirsch, A. Miele, J. Stoer, K. H. Well (eds.): Optimal Control — Calculus of Variations, Optimal Control Theory and Numerical Methods, International Series of Numerical Mathematics **111** (Basel: Birkhäuser, 1993) 129-143.

[9] Stryk, O. von: Numerische Lösung optimaler Steuerungsprobleme: Diskretisierung, Parameteroptimierung und Berechnung der adjungierten Variablen. Fortschritt-Berichte VDI, Reihe 8, Nr. 441 (Düsseldorf: VDI-Verlag, 1995).