A decomposition approach for optimal control problems with integer inner point state constraints

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A large class of optimal control problems for hybrid dynamic systems can be formulated as mixed-integer optimal control problems (MIOCPs). A decomposition approach is suggested to solve a special subclass of MIOCPs with mixed integer inner point state constraints. It is the intrinsic combinatorial complexity of the discrete variables in addition to the high nonlinearity of the continuous optimal control problem that forms the challenges in the theoretical and numerical solution of MIOCPs.

During the solution procedure the problem is decomposed at the inner time points into a multiphase problem with mixed integer boundary constraints and phase transitions at unknown switching points. Due to a discretization of the state space at the switching points the problem can be decoupled into a family of continuous optimal control problems (OCPs) and a problem similar to the asymmetric group traveling salesman problem (AGTSP).

The OCPs are transcribed by direct collocation to large-scale nonlinear programming problems, which are solved efficiently by an advanced SQP method. The results are used as weights for the edges of the graph of the corresponding TSP-like problem, which is solved by a Branch-and-Cut-and-Price (BCP) algorithm.

The proposed approach is applied to a hybrid optimal control benchmark problem for a motorized traveling salesman.

1 Decomposition approach

A MIOCP with inner point constraints can be defined as minimizing the performance index

$$J[\mathbf{u},\mathbf{v},\mathbf{z}] = \sum_{i=0,\dots,n_c} \varphi\left(\mathbf{x}(t_i^c), t_i^c\right) + \int_{[t_0,t_f]} L(\mathbf{x}(t),\mathbf{u}(t),\mathbf{v}(t),t)\,\mathrm{d}t \tag{1}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),\mathbf{v}(t)), \quad t_0 \le t \le t_f, \tag{2}$$

$$\mathbf{c_{min}} \le \mathbf{c}(\mathbf{x}(t),\mathbf{u}(t),\mathbf{v}(t)) \le \mathbf{c_{max}}, \tag{3}$$

$$0 = \mathbf{r}(\mathbf{x}(t_i^c),\mathbf{z},t_i^c), \quad t_0 = t_0^c \le t_1^c \le \cdots \le t_{n_c}^c = t_f, \tag{4}$$

with $\mathbf{u}(t) \in \mathcal{U} \subset \mathbb{R}^{n_u},\ \mathbf{v}(t) \in \mathcal{V} \subset \mathbb{Z}^{n_v},\ \mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^{n_x},\ \mathbf{z} \in \{\mathbf{z}_1,\dots,\mathbf{z}_{n_w}\} \subset \mathbb{Z}^{n_z \times n_w}$ for all $t \in [t_0, t_f]$ [1]. In this work it is assumed, that no integer control $\mathbf{v}(t)$ is present. Further a transformation of the mixed integer problem into a mixed binary problem (MBOCP), which is linear in the binary parameters is done. Therefore equation (4) is replaced by

$$0 = \sum_{j=1,\dots,n_w} w_{ij}\mathbf{r}(\mathbf{x}(t_i^c),\mathbf{z}_j,t_i^c) =: \mathbf{R}(\mathbf{x}(t_i^c),t_i^c)\mathbf{w}_i,\ \ 1 = \sum_{j=1,\dots,n_w} w_{ij},\ \ \mathbf{w}_i \in \{0,1\}^{n_w},\ \ i = 0,\dots,n_c. \tag{5}$$

This transformation can also be done analogue for other constraints depending on integer parameters or variables. Thus a mixed integer problem can always be transformed into a mixed binary problem, which is linear in the binary variables.

To find a solution for the MBOCP one could solve all problems related to feasible $\mathbf{w}$ and choose the one with lowest cost. This approach may work for small numbers n_w, but it is not appropriate if n_w grows. Therefore other approaches are needed.

At the inner points t_i^c the MBOCP can be cut into phases. Thus the same inner point constraint must hold at the end of one and at the beginning of the next phase. In addition phase transitions are introduced, which assure the continuity of the states. A closer look to one of these phases shows, that it is impossible to solve this phase independent of the others, because of the transitions at the switching points t_i^c. If the state at initial and final time of the phase would be fixed and $\mathbf{w}$ would be given the single phase OCP could be solved. If the states are discretized $(\mathbf{x}_{i,0},\ \dots \mathbf{x}_{i,n_d})$ at the switching points t_i^c not violating the inner point constraints (Fig. 1l), the single phase OCP can be solved for each pair of initial values $(\mathbf{w}_k, \mathbf{x}_{i,l}) =: p_a$ and final values $(\mathbf{w}_o, \mathbf{x}_{i+1,m}) =: p_b$ independent of the other phases. Thus for each of these point to point trajectories an optimal cost c_{ab} can be evaluated numerically. This is done by direct collocation. Hereby the time interval is discretized. The state variables are approximated on this grid by continuous piecewise cubic and the control variables by non-continuous piecewise linear functions. The collocation constraints are fulfilled at Gaussian points, the other constraints are fulfilled pointwise. So the OCPs are transcribed into large sparse NLPs, which are solved by the state of the art SQP-solver SNOPT (P. E. Gill et al.).

Each discrete point p_i can be interpreted as a vertex in a graph $G(V, E)$ (Fig. 1m), and all edges $ij \in E$ between two vertices p_i and p_j, $i, j \in V$ can be adopted with the optimal cost of the corresponding single phase OCP as its weight. If the

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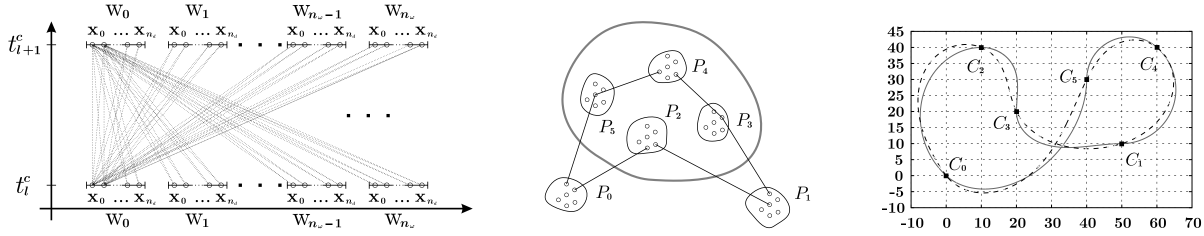

Fig. 1 Solution steps of the decomposition approach.

OCP can not be solved the weight can be set to infinity. An approximation of the MBOCP is a valid path in this graph. For the edges binary variables, which are equal to one if the edge is part of the path and zero otherwise, are introduced to formulate a linear combinatorial problem, which is numerically solved by the BCP algorithm SYMPHONY (T. K. Ralphs).

2 Motorized Traveling Salesman Problem (MTSP)

As an example the hybrid benchmark-problem of the motorized traveling salesman [1] is solved. The task of the salesman is to find an acceleration and braking force $\bar{\beta}(t)$ and a steering angle velocity $\bar{\gamma}(t)$ to control the car and in addition an order $\bar{\mathbf{w}}_i$ of the cities to be visited by him to minimize the overall traveling time. This problem is easily scalable in continuous (i.e. car model) and discrete (i.e. number of cities) complexity. Here the car is assumed to be a simple point mass located in $(x(t), y(t))$ with velocity $v(t)$ and driving direction $\alpha(t)$ and only $n_c = 6$ cities are considered. Thus the problem reads as

$$\min\ J[\mathbf{u},\mathbf{w}] \;=\; t_f + 2\int_{[t_0,t_f]} (\beta(t)^2 + \gamma(t)^2)\mathrm{d}t \tag{6}$$

$$\text{s.t. } (\dot{x}(t), \dot{y}(t), \dot{v}(t), \dot{\alpha}(t))^T \;=\; (v(t)\cos(\alpha(t)), v(t)\sin(\alpha(t)), \beta(t), \gamma(t))^T \tag{7}$$

$$(x(0), y(0), v(0), \alpha(0))^T \;=\; x(t_f), y(t_f), v(t_f), \alpha(t_f))^T \tag{8}$$

$$0 \le v(t) \le 10, \quad |\beta(t)| \le 2, \quad |\gamma(t)| \le 1. \tag{9}$$

$$(x(t_i^c), y(t_i^c))^T - \mathbf{C}\mathbf{w}_i \;=\; 0, \quad \sum_{i=0,\dots,n_c} \mathbf{w}_i \;=\; (1,\dots,1)^T. \tag{10}$$

Where $\mathbf{C} = (C_0, \dots, C_{n_c}) \in \mathbb{R}^{2\times n_c}$ includes the cities locations and $\mathbf{w}_i \in \{0,1\}^{n_c}$ is an unit vector. For decomposition the states $v(t_i^c)$ and $\alpha(t_i^c)$ are discretized into overall $n_d = 40$ (5 for velocity, 8 for angle) points. Neglecting any symmetries $n_c(n_c - 1)n_d^2 = 1200$ similar single phase OCPs have to be solved, which leads to a computation time of about 3 hours (0.15s per problem on a 2GHz PC). With the achieved costs c_{ij} the resulting combinatorial problem is an asymmetric group TSP [2].

$$\min \sum_{ij\in E} \omega_{ij} c_{ij} \tag{11}$$

$$\text{sc.} \sum_{ij\in\eta^+(\{l\},P_k)} \omega_{ij} \;\le\; \nu_{l,k}, \quad \sum_{ij\in\eta^-(\{l\},P_k)} \omega_{ij} \;\le\; \nu_{l,k}, \quad \forall\, l\in P_k,\ k\in\{1,\dots,n_c\}, \tag{12}$$

$$\sum_{l\in P_k;} \nu_{l,k} \;=\; 1, \quad \sum_{ij\in E(P_k)} \omega_{ij} \;=\; 0, \quad \forall\, k\in\{1,\dots,n_c\}, \tag{13}$$

$$\sum_{ij\in E(Q)} \omega_{ij} \;\le\; |I| - 1, \quad \forall\, Q = \bigcup_{k\in I} P_k,\ I\subset\{1,\dots,n_c\}, \tag{14}$$

$$\omega_{ij} \;\in\; \{0,1\}, \quad \forall\, ij\in E,\ \nu_k \;\in\; \{0,1\},\ \forall\, k\in\{1,\dots,n_c\} \tag{15}$$

with $\eta^+(Q,R) := \{ij \in E \mid i\in Q,\ j\in V\setminus R\}$, $\eta^-(Q,R) := \{ij\in E \mid i\in V\setminus R,\ j\in Q\}$ and $E(Q) := \{ij \in E \mid i,j\in Q\}$. Using BCP the problem could be solved in 12 minutes. The cost of the discretized journey (Fig. 1r) are 40.98. Solving the continuous problem for this fixed order of cities (Fig. 1r, dotted line) leads to a cost of 37.87. The difference is quite large. Thus a refinement or a linear interpolation of the cost on the state discretization could improve the results of the presented decomposition approach.

References

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