This is a preprint of the paper that appeared in:
Proc. 15th IFAC World Congress on Automatic Control, Barcelona, Spain, July 21-26 (Elsevier Science, 2002) 1636-1641.

INCREASING STABILITY IN DYNAMIC GAITS USING NUMERICAL OPTIMIZATION

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Abstract: Optimal gait planning is applied in this work to the problem of improving stability in quadruped locomotion. In many settings, it is desired to operate legged machines at high performance levels where rapid velocities and a changing environment make stability of utmost concern. Since gait planning still remains a vital component of legged system control design, an efficient method of determining periodic paths is presented which optimize a dynamic stability criterion. Efficient recursive multibody algorithms are used with numerical optimal control software to solve the minimax performance stability criteria.

Keywords: walking, path planning, dynamic stability, robot dynamics, optimization problems, nonlinear programming, numerical methods

1. INTRODUCTION

Legged locomotion systems are characterized by very nonlinear dynamics of high dimension, and it is a complex task to generate and control stable gaits for such systems. There are two general gait classifications, namely static gaits and dynamic gaits. While static gaits have only a kinematic measure of stability and can result in very slow forward movement, dynamic gaits have a much wider range of motion. Dynamic gaits are consequently much harder to control, and their stability harder to measure. For this purpose, the tools of numerical optimization can provide an objective means for evaluating a given stability measure. In addition, on-line and calculational considerations must be made. Numerical optimization techniques which have previously been used for periodic gait generation are implemented here for this purpose.

Static stability can be measured from the system's kinematic configuration. As introduced by (McGhee and Frank, 1968), the static stability margin (SM) is the shortest distance between the projected center of mass of the system and the boundaries of the support polygon formed by the convex hull of the supporting feet. This can also be divided by the center of mass height (SI = SM/h) to additionally penalize a larger potential energy. A significant improvement to this index is the energy stability margin from (Messuri and Klein, 1985). This stability index measures the energy required to tip a system over one of the edges in its support polygon or equivalently the change in potential energy to rotate the system, assuming the system behaves as an inverted pendulum, to bring the center of mass over the support boundary. The principal defects of static stability measures, though, are that they do not take into account velocity, inertial effects, and the influence of the swinging legs and their future ground contact.

For many-legged systems, primarily those with six or more legs, static stability is often sufficient as gaits may be selected such that three or more legs are in contact with the ground. For quadrupeds, this is particularly restrictive so that dynamic stability is of greater importance. Any attempt to exploit the increased dexterity of biped or quadruped systems or to emulate their biological counterparts requires operating in regimes where static stability is not feasible. A description of the popular method of quantifying dynamic stability using the zero moment point (ZMP) may be found in (Vukobratović *et al.*, 1990) which remains a widely used measure. There it is pointed out that two very different forms of measuring dynamical stability correspond to measuring either postural or gait stability. The ZMP belongs to the class of postural stability measures which ensure that the system configuration remain within a specified region of state space, yet as argued in (Goswami, 1999) this measure has its defects since the ZMP is constrained to lie within the ground support polygon. As the system becomes unstable, this point will lie on the boundary of the support polygon and thus provide little information as to the amount of instability. Additionally, many interesting gaits exist for quadrupeds which have only one-dimensional support polygons. Further work in (Koo and Yoon, 1999) used the angular momentum to define a stability index about the support edges. In contrast to previous approaches, this method can provide both directional information and a reference stability quantity that can be used for measuring postural stability/instability while considering system kinetics. Though not a rigorous measure for the stability of a dynamical system, such a measure provides a means to monitor the stability of walking systems *on-line* and additionally can treat non-periodic gaits, a necessity when the terrain is rough or there exists obstacles. For this reason, this is the principal measure used later in the numerical investigations.

A measure of gait stability when considering periodic gaits employs Floquet theory from nonlinear system theory which can be used to investigate the local stability of critical points of discrete maps. A stability analysis of human walking data was conducted in (Hurmuzlu and Basdogan, 1994) using this approach. In recent work (Mombaur *et al.*, 2001), the maximum eigenvalue of the monodromy matrix, the Jacobian of the periodic map, is minimized to compute open-loop stable passive trajectories for a simple planar biped. These methods still remain difficult to apply to a full-dimensional quadruped model due to the nonconvexity of the optimization and are not suited for on-line calculations, gait transitions, or non-periodic gaits.

We explore in this paper an optimization approach based on a discretization of the control problem and its subsequent formulation as a nonlinear optimization problem which has successfully been applied for gait planning in bipeds without model simplifications in two dimensions (Hardt *et al.*, 1998) and for quadrupeds in (Hardt and von Stryk, 2000). The measure of dynamic stability employed by (Koo and Yoon, 1999) for legged systems is studied here using a complete three-dimensional model of the quadruped. The system is optimized using a minimax objective to maximize the minimum stability during the gait cycle. The calculations are made more efficient with the use of a reduced multibody dynamics algorithm without making any model simplifications and the application of an efficient optimal control method based on a direct collocation transcription and a sparse sequential quadratic programming algorithm.

2. QUADRUPED LOCOMOTION

2.1 *Quadruped Dynamic Model*

Table 1. Physical data of quadruped model

	Body	Thigh	Shank
m (kg)	37.6	3.7	2.15
I_{xx} ($kg-m^2$)	1.60	0.003	0.001
I_{yy} ($kg-m^2$)	3.40	0.03	0.01
I_{zz} ($kg-m^2$)	4.30	0.03	0.01
length (m)	0.72	0.29	0.29
distance to CM (m)	0	0.146	0.032

The quadruped model considered here consists of a 9-link tree-structured multibody system with a central torso and 4 two-link legs. Each leg contains a 2 *dof* universal joint in the hip and a 1 *dof* rotational joint in the knee. The links are modeled with a uniform density of mass. The physical data used in the experiments can be found in Table 1 and are based on the quadruped walking machine Warp1 (Ridderström *et al.*, 2000). A minimum set of coordinates consists of 36 continuous states $(\mathbf{q}(t), \mathbf{v}(t))$ and 12 control variables $\mathbf{u}(t)$ which include the Bryant Euler angles and global position of the torso body, the angular and linear velocity of the torso body, and the three angles and their velocities for each leg. The generalized velocities $\mathbf{v}$ must not necessarily equal the time derivative of the generalized positions $\mathbf{q}$ as is the case with the torso orientation where $\mathbf{v}$ contains the angular velocity of the torso. When evaluating the time derivative of the Bryant angles, a kinematical transformation must first be performed from the torso angular velocity. The remaining velocities though are equal to $\dot{\mathbf{q}}$.

The equations of motion are those for a rigid, multibody system experiencing contact forces

$$\begin{aligned} \dot{\mathbf{q}} &= k(\mathbf{v}) \\ \dot{\mathbf{v}} &= \mathcal{M}(\mathbf{q})^{-1}\Big(\mathbf{u} - \mathcal{C}(\mathbf{q},\mathbf{v}) - \mathcal{G}(\mathbf{q}) + J_c(\mathbf{q})^T \mathbf{f}_c\Big) \end{aligned} \quad (1)$$

where k is a kinematic transformation, $\mathcal{M}$ is the square, positive-definite mass-inertia matrix, $\mathcal{C}$ is the vector of Coriolis and centrifugal forces, $\mathcal{G}$ is a vector of gravitational forces, $\mathbf{u}(t)$ are the control input functions, J_c is the constraint Jacobian, and f_c is the constraint force. The ground contact constraints represent holonomic constraints on the system $g_c(\mathbf{q}) = 0$. The constraint Jacobian has the form $J_c = \frac{\partial g_c}{\partial \mathbf{q}}$ while the vector $\mathbf{f}_c$ may also be interpreted as Lagrange multipliers.

The multibody equations of motion are computed using recursive $\mathcal{O}(N)$ multibody algorithms based on the work in (Rodriguez *et al.*, 1991) and which have evolved into an object-oriented multi-faceted computer package (Helm *et al.*, 2002). It is well-known that recursive algorithms are more efficient in their dynamics calculations for systems with more than 7 or 8 degrees of freedom than alternative approaches which involve assembling the entire mass-inertia matrix. Other multibody dynamics algorithms are perhaps better suited for smaller systems or where they may exist a larger number of interior kinematic closed-chains.

2.2 *Gait Classification and Constraints*

When formulating the periodic constraints and the constraints for jumps in the state velocities at the collision events during a periodic gait, one must know in advance the discrete sequence of such events. For a biped, this is not a problem as one need only distinguish between walking and running. The range of different quadruped gaits, however, can be quite large. The problem of searching over all possible gaits in a gait optimization problem has not yet been completely solved (Hardt and von Stryk, 2000) yet biological studies can provide a good indication as to which quadrupedal gaits are the most efficient for different forward velocities (Alexander, 1984). A numerical advantage for considering gaits with the left and right legs of a pair with equal duty factors is that the problem can be completely formulated within half a gait cycle. The fixed duty factor of $\beta = 0.5$ is used in the numerical investigations. All other gait parameters are optimized.

The inelastic collision of a leg with the ground introduces an impulsive force to the system which in turn produces a discontinuous jump in the generalized velocities. The jumps in the velocities may similarly be computed with efficient $O(N)$ recursive algorithms (Hardt *et al.*, 1998).

2.3 *Reduced Dynamics*

The dynamics for walking machines are particularly well-suited for using reduced-dynamics algorithms free of algebraic constraints due to the relative ease with which the inverse kinematics problem may be solved in the case of a leg experiencing contact. For example, using the quadruped model previously described with a 2 *dof* hip joint and 1 *dof* knee joint, all of the states corresponding to a leg experiencing contact are dependent upon the torso states. They may be solved for uniquely given merely the hip position, hip linear and angular velocity, and its ground contact position.

This approach, also known as coordinate partitioning (Ascher and Petzold, 1998), solves for a reduced set of independent states from (1) thus converting the DAE contact system into an ODE system of minimal size. This method requires solving the inverse kinematics problem for the dependent states which, in the case of legged systems, are generally the contact leg states. For most leg configurations, this problem is easily solved using knowledge of the relative hip and foot contact locations. This approach is not an approximation to the dynamics but rather a more efficient computational method.

Define a change of variables for the position states

$$\mathbf{r}_1 = Z\mathbf{q}, \quad \mathbf{r}_2 = i(\mathbf{r}_1)\ . \tag{2}$$

The transformation $Z \in \mathbb{R}^{(N-l)xN}$ is a constant matrix consisting of unit vectors or $\mathbf{0}$ chosen such that $\mathbf{r_1}$ are the independent states. The closed-form inverse kinematic solution for the dependent states is calculated with the function $i(\cdot)$. Partitioning the constraint velocity equation $J_c\mathbf{v} = 0$ with respect to the independent $\mathbf{v}_1$ and dependent velocity states $\mathbf{v}_2$, $J_{c,1}\mathbf{v_1} + J_{c,2}\mathbf{v_2} = 0$, provides a change of variables for the velocity states

$$\mathbf{s}_1 = Z\mathbf{v}, \quad \mathbf{s}_2 = -J_{c,2}^{-1}J_{c,1}\mathbf{s}_1\ . \tag{3}$$

Introducing these quantities into (1) and multiplying by Z then gives an ODE of size $(N-l)$

$$\dot{\mathbf{s}}_\mathbf{1} = Z\mathcal{M}^{-1}\left(\mathbf{Ru} - \mathcal{C} - \mathcal{G} + J_c^T\mathbf{f}_c\right)\ . \tag{4}$$

The principal advantage of this approach is that one may calculate the equations of motion using the full state, yet one need only perform the optimization on the reduced dimensional state. The state must then be monitored such that it remain within a well-defined region of the state space. In Section 3.1, where the optimal amble gait is investigated for a quadruped, there are always two legs in contact. As a result, instead of the full 36 states $(\mathbf{q},\mathbf{v})$, 24 states describe the system.

The state equations (1) are of reduced dimension and of variable structure since the form of the reduced state will depend on the contact condition. We introduce an additional discrete state variable c_i associated with each leg that describes the ground contact condition of the i^{th} leg at time t:

$$c_i : [0,t_f] \to \{1,2\},\ i = 1,\ldots,4$$
$$c_i(t) = \begin{cases} 1, & \text{no contact or swinging phase} \\ 2, & \text{fixed contact phase} \end{cases} \tag{5}$$

Then the dynamics of the legged system have the general variable structure form:

$$\dot{\mathbf{v}}(t) = \begin{cases} \mathbf{f}^1(\mathbf{q}(t),\mathbf{v}(t),\mathbf{c}(t),\mathbf{u}(t),\mathbf{p},t),\ t \in [t_0,t_{S,1}], \\ \mathbf{f}^k(\mathbf{q}(t),\mathbf{v}(t),\mathbf{c}(t),\mathbf{u}(t),\mathbf{p},t), \\ \qquad t \in [t_{S,k-1},t_{S,k}], \quad k = 2,\ldots,m-1 \\ \mathbf{f}^m(\mathbf{q}(t),\mathbf{v}(t),\mathbf{c}(t),\mathbf{u}(t),\mathbf{p},t),\ t \in [t_{S,m-1},t_f], \end{cases} \tag{6}$$

where the discrete state $\mathbf{c}(t)$ is constant in each phase $[t_{S,i},t_{S,i+1}]$. In Section 3.1, where the optimal amble gait is investigated for a quadruped, there are always two legs in contact. As a result, instead of the full 36 states $(\mathbf{q},\mathbf{v})$, 24 states can describe the system.

2.4 *Performance Specifications and Dynamic Stability*

In (Koo and Yoon, 1999), an approach was presented for measuring the dynamic stability of a quadruped based on the angular momentum. The main features of this measure is that it is an instantaneous measure which can be applied at any moment during the gait unlike those based on the Floquet multipliers and which are only applicable to periodic gaits. It differs from previous approaches in that the momentum of the swing legs are also taken into consideration. It is however limited to gaits with at least two legs in contact with the ground. Various simulations of dynamics gaits were conducted in (Koo and Yoon, 1999), and it was demonstrated how the stability index can effectively be used in closed-loop in addition to indicating how several gait parameters influence gait stability. The work presented here involves the generation of dynamically stable gaits and will be primarily based on the stability index of (Koo and Yoon, 1999). The planning of gaits is an important part in the control design of dynamic gaits, particularly when gaits are desired which are far from being statically stable as no closed-loop controllers exist which can guarantee dynamic stability under all conditions.

The gait stability measure in (Koo and Yoon, 1999) is computed as

$$S_H = \min\{S_H^l,\ l = 1,\ldots,n_l\} \tag{7}$$

where n_l is the number of edges in the support polygon. The stability values S_H^l for each edge depend on whether the edge is a diagonal or non-diagonal edge. The linear and angular momentum of the system about the system center of gravity **CG** are denoted as $\mathbf{L}_{CG}$ and $\mathbf{H}_{CG}$ respectively. These values can be used to compute the the angular momentum about a point $\mathbf{P}$ on the ground using the transformation $\mathbf{H}_P = \mathbf{H}_{CG} + \mathbf{r}_{P,CG} \times \mathbf{L}_{CG}$ where $\mathbf{r}_{P,CG}$ is the vector from $\mathbf{P}$ to **CG**. When $\mathbf{P}$ lies on the edge connecting two support legs, the rotational tendency about that edge is

$$H_l = (\mathbf{H}_{CG} + \mathbf{r}_{P,CG} \times \mathbf{L}_{CG}) \cdot \hat{\mathbf{e}}_l \tag{8}$$

where $\hat{\mathbf{e}}_l$ is the unit vector along edge l. The reference angular momentum H_l^{ref} about edge l is defined as the minimum angular momentum to tip over the edge if the system were an inverted pendulum

$$H_l^{ref} = (\mathbf{r}_{l,CG} \times m_{total}\mathbf{v}_{ref}) \cdot \hat{\mathbf{e}}_l\ . \tag{9}$$

Here $\mathbf{r}_{l,CG}$ is the orthogonal vector from the edge l to CG and m_{total} is the total system mass. The reference velocity vector $\mathbf{v}_{ref}$ is computed from the kinetic energy required to attain the higher potential energy at which the system CG would lie above edge l. The difference in potential energy from the current position to the point of instability is the same as the energy stability margin given in (Messuri and Klein, 1985)

$$m_{total}(g\|\mathbf{r}_{l,CG}\| \cos\psi_l - \mathbf{g}^T \mathbf{x}_{CG}) \tag{10}$$

where $\mathbf{g} = [0\ 0\ g]^T$ is the vector of gravitational force. Then for a non-diagonal edge $S_H^l = H_l^{ref} - H_l$.

In the case of a diagonal edge, the maximum angular momentum is additionally defined:

$$H_l^{max} = (\mathbf{r}_{l,CG} \times m_{total}\mathbf{v}_{tip}^{max}) \cdot \hat{\mathbf{e}}_l \tag{11}$$

where $\mathbf{v}_{tip}^{max}$ is the maximum velocity vector of the swing leg's tip. If only two legs are in contact, then when the CG has not yet crossed the diagonal edge then $S_H^l = \min\left(H_l - H_l^{ref}, H_l^{max} - H_l\right)$, and if it has crossed $S_H^l = H_l^{max} - H_l$. In the case of a third supporting leg behind the diagonal edge $S_H^l = H_l^{max} - H_l$, and if the supporting leg lies in front then $S_H^l = H_l^{ref} - H_l$.

An overall gait stability measure may also be constructed from the above when dealing with a periodic gait by defining it to be the minimum value of S_H over the entire gait cycle or its integrated value. As stability is more of a risk criterion than a cost criterion, the minimum value formulation is used in the subsequent analysis.

3. NUMERICAL OPTIMIZATION

It is demonstrated how the optimal gait planning problem of maximizing the minimum value of the stability index can be framed as a nonlinear programming problem which may then be solved using efficient optimization software. In addition, current computing technology makes the optimization of a complete three-dimensional dynamical model of a quadruped possible within a reasonable amount of time.

The method of sparse direct collocation (von Stryk, 1995) is used to approximate the states $\mathbf{x} = (\mathbf{q},\mathbf{v})$ and controls $\mathbf{u}$ of the optimal control problem along the subintervals $t \in [t_k^j, t_{k+1}^j]$ of a grid $t_{S,j-1} = t_1^j < t_2^j < \cdots < t_{n_{G,j}}^j = t_{S,j}$ in each phase ,

$$\begin{aligned} \tilde{\mathbf{u}}_{app}(t) &= \beta(\hat{\mathbf{u}}(t_k^j), \hat{\mathbf{u}}(t_k^{j+1})), & \beta - \text{linear} \\ \tilde{\mathbf{x}}_{app}(t) &= \alpha(\hat{\mathbf{x}}(t_k^j), \hat{\mathbf{x}}(t_k^{j+1}), \mathbf{f}_k^j, \mathbf{f}_{k+1}^j), & \alpha - \text{cubic} \end{aligned} \tag{12}$$

where $\mathbf{f}_k^j = \mathbf{f}^j(\hat{\mathbf{x}}(t_k^j), \hat{\mathbf{u}}(t_k^j), \mathbf{p}, t_k^j)$. The infinite-dimensional optimal control problem is thereby converted to a finite-dimensional constrained nonlinear program containing the unknown values for $\mathbf{x},\mathbf{u},\mathbf{p},t_{S,i},t_f$. The problem is then solved using an SQP-based optimization code for sparse systems SNOPT (Gill *et al.*, 1998). It is equipped to handle general nonlinear equality and inequality constraints on the states and controls including magnitude bounds, multiple phases with switching dynamics, jumps in the states and controls, and objectives with continuous and discrete costs (von Stryk and Glocker, 2001). The method is thus equipped to handle the complexities of the walking problem: unknown liftoff times, different ground contact combinations for the legs, discontinuous states at collision times of the legs with the ground, switching dynamics, and actuation limits.

The desired performance is the min-max criterion:

$$J[\mathbf{u}] = \min_{0 \le t \le t_f} S_H(t) \longrightarrow \max! \tag{13}$$

Similar to the optimal windshear control problem solved in (Bulirsch *et al.*, 1991), by adding an additional control parameter p_1 to the problem, the min-max problem may be transformed to a standard form of Mayer-type objective:

$$p_1 := \min_{0 \le t \le t_f} S_H(t)$$

where an additional inequality constraint is needed,

$$S_H(t) - p_1 \ge 0, \quad 0 \le t \le t_f$$

and the transformed objective becomes $\tilde{J}[\mathbf{u}, \mathbf{p}] = -p_1$.

This objective maximizes the minimally attained level of postural stability during the dynamic gait. The gait stability though must be measured by other means. A post-processing of the calculated solutions is the most feasible manner of accomplishing this. Nonlinear dynamical systems theory dictates that the gait stability of periodic systems may be determined by the eigenvalues (Floquet multipliers) of the sensitivity matrix (monodromy matrix) between the final state values $\mathbf{x}(t_f)$ and the initial state values $\mathbf{x}(0)$ of the optimal control problem. The eigenvalues are equivalent to the linearized Poincaré map and indicate asymptotic stability when they are less than unity. The matrix calculation however requires an accurate calculation of the sensitivities to the contact dynamics, constraints, and boundary conditions and thus represents a computational hurdle. This important validative approach will be incorporated into future work.

3.1 *Numerical Experiments*

Fig. 1. The quadruped walking machine Warp1 of (Ridderström *et al.*, 2000).

The numerical experiments were conducted using the model data from Table 1 based on the quadruped walking machine Warp1 (Ridderström *et al.*, 2000) (Figure 1). The optimization requires a set of starting values (rather than an arbitrary set) to begin the iteration procedure. Due to the complexity of the model, a reasonable set of starting values is required for the optimization to converge to a solution. This can be done in a number of ways. One is to use heuristic arguments to determine the state position paths and feet placement locations, desired forward velocities to determine the state velocities, and desired ground forces with inverse dynamics to calculate a set of control inputs. An alternative is to calculate a series of optimal solutions for subproblems. The latter approach was the one taken here where first the problem was solved in two dimensions with most parameters fixed, then all constraints were gradually relaxed using the previous solution to start the subsequent problem. Optimization run times for a single subproblem ranged from 5 to 20 minutes on a Pentium III, 900 MHz PC.

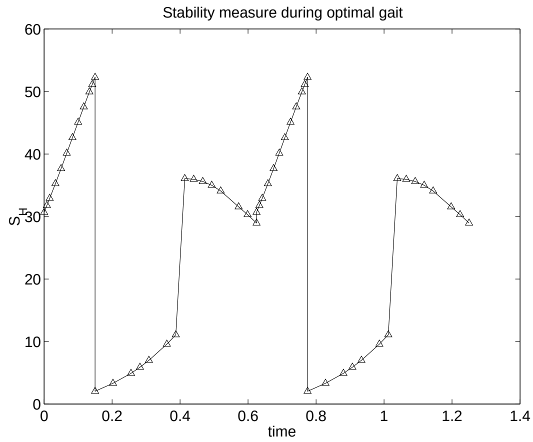

Fig. 2. Stability index S_H for optimized gait of a quadruped moving forward at 0.67 m/s and duty factor $\beta = 0.5$ based on a full three-dimensional dynamical model.

The solution displayed is for a desired forward velocity of 40 *m/min* or 0.67 *m/s*. The optimal gait stride is 0.416 *m* and the gait period is 1.25 seconds. The only gait parameter fixed during the optimization is the duty factor of $\beta = 0.5$ while other parameters such as average forward velocity, relative phases or step size have been optimized as well. Figure 2 shows the evolution of the optimized stability index S_H over one gait period. A negative value of S_H indicates an unstable configuration while the more positive S_H is, the more stable the system. The large variations in the index are caused by the changing of support legs in the robot. The gait displayed is an amble gait between walking and running with the legs having a duty factor of $\beta = 0.5$ which is a demanding, fast-moving gait. The order of support leg order is (LF-LB, LR-RR, RF-RR, RF-LR: LF=left front, RR=right rear) so that the system alternates between having two support legs on one side and diagonal support legs. The steepest drop in S_H occurs when the system switches from a side pair of support legs to a diagonal pair. At that point the angular momentum of the system about the diagonal edge is slightly greater than the required angular momentum for the **CG** to "roll over" the diagonal edge and not fall backward. A conservative value of 2 *m/sec* was chosen for the attainable velocity of the swing leg tip $\mathbf{v}_{tip}^{max}$.

Ongoing numerical investigations test these methods for different gait classifications, combine them with trajectory tracking control laws, compare with alternative energy-based performances, validate them with full 3D dynamics simulations including different disturbances such as slipping and tripping before final implementation. It is planned to validate the numerical results in experiments in cooperation with KTH Stockholm (Ridderström *et al.*, 2000).

3.2 *Experimental Results*

Fig. 3. The four-legged Sony robots used in our group for robot soccer competition (RoboCup).

Another immediate application of the current work is in the design of fast and stable walking strategies of the four-legged Sony robots used for RoboCup soccer competition in our group (Figure 3). Here, energy efficiency is not important since the robot's battery can be replaced frequently, but fast and stable motions are most important for the performance in a competition. Compared with the previous four-legged robot dynamic model a main difference is not the overall size but the relatively heavy head which also must be considered.

4. CONCLUSION

The present work extends recent results in optimal gait planning to the important case of maximum gait stability. In many settings, this criterion can be of primary importance such as when obstacles must be avoided, energy concerns are less of a problem, or moving at a high velocity is desired. The investigated dynamic stability criterion is well suited to a changing environment and on-line stability assessment for closed-loop control design. Efficient recursive multibody algorithms combined with efficient numerical optimal control software solve the minimax performance stability criteria.

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