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Numerical mixed-integer optimal control and motorized traveling salesmen problems

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ABSTRACT: *A general approach for the numerical solution of hybrid, mixed-integer optimal control problems is presented. In an outer level iteration a branch-and-bound procedure is applied to search the entire feasible discrete variable space. An inner level iteration contains for each actual value of the discrete variable a continuous nonlinear optimal control problem with its nonlinear dynamics defined in multiple phases and phase transitions occuring at unknown switching points (events) which must be solved numerically subject to nonlinear constraints. For this purpose, a robust and efficient direct collocation method is employed that parameterizes both the continuous state and control variables and exploits the sparse structure in the resulting nonlinearly constrained optimization problems. The proposed approach is successfully applied to two new hybrid optimal control benchmark problems for a motorized traveling salesman and for a team of two cooperating, motorized salesmen.*

RÉSUMÉ: *Cet article présente une méthode générale pour la solution numérique de problèmes hybrides de commande optimale. Dans l' itération extérieure, une méthode de type "Branch and Bound" est appliquée pour explorer tout l' espace admissible de variables discrètes. Dans l'itération intérieure des problèmes de commande optimale doivent être résolus pour chaque valeur des variables discrètes. La dynamique non-linéaire de ces problèmes de commande optimale est définie dans plusieurs phases et instants de commutation, elle est limitée par des inéquations non-linéaires. La méthode proposée est appliquée à deux nouveaux cas d'études portant sur la commande optimale d'un voyageur de commerce motorisé et d'une équipe faisant coopérer deux voyageurs de commerce motorisés.*

KEY WORDS: *nonlinear hybrid dynamical systems, mixed-integer optimal control, branch-and-bound, sparse direct collocation, motorized traveling salesman.*

MOTS–CLÉS: *systèmes dynamiques hybrides nonlinéaires, commande optimale "entiers mixtes", branch and bound, collocation directe et adaptée, voyageur de commerce motorisé.*

1. Mixed-integer optimal control problem

Optimal control problems of hybrid dynamical systems occur in many applications, e. g., in process engineering [ADJ 99, ALL 97, BEM 99], underactuated robotics [BUS 00, HAR 00b], and walking robots [HAR 00a]. Here, we consider a nonlinear dynamical system defined in n_c+1 phases $[t_i^c, t_{i+1}^c]$, $i = 0, \ldots, n_c$, (cf. Fig. 1)

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}_{(i)}(\mathbf{x}(t), \mathbf{u}(t), \omega, t), \;\; t_i^c \le t < t_{i+1}^c, \qquad [1]$$

with a piecewise continuously differentiable state variable $\mathbf{x} : [0, t_f] \to \mathbb{R}^{n_x}$, a piecewise continuous control variable $\mathbf{u} : [0, t_f] \to \mathbb{R}^{n_u}$, and a binary control vector $\omega \in \{0,1\}^{n_\omega}$. The transition from phase i to phase $i+1$ takes place at the usually unknown event time t_i^c, $i = 1, \ldots, n_c$, i. e., a switching point. The mixed-integer optimal control problem (MIOCP) is defined as minimizing the real-valued, hybrid performance index

$$J[\mathbf{u}, \omega] \;=\; \sum_{i=1}^{n_c+1} \left(\varphi_{(i)} \left(\mathbf{x}(t_i^c - 0), \mathbf{x}(t_i^c + 0), \omega, t_i^c \right) + \int_{t_{i-1}^c}^{t_i^c} L_{(i)}(\mathbf{x}(t), \mathbf{u}(t), \omega, t) \, \mathrm{d}t \right) [2]$$

with respect to the continuous control variable $\mathbf{u}$ and the discrete control vector ω and subject to state and control variable inequality and equality constraints

$$0 \;\le\; g_{(i),j}(\mathbf{x}(t), \mathbf{u}(t), \omega, t), \quad j = 1, \ldots, n_{g(i)}, \qquad [3]$$

$$0 \;=\; h_{(i),j}(\mathbf{x}(t), \mathbf{u}(t), \omega, t), \quad j = 1, \ldots, n_{h(i)}. \qquad [4]$$

Number and type of constraints differ from one phase to the other. Furthermore, at initial, final, and switching times implicit and explicit switching conditions may hold

$$0 \;=\; r_{(0),j}(\mathbf{x}(0), \mathbf{x}(t_f), \omega, t_f), \;\; j = 1, \ldots, n_{r_{(0)}}, \qquad [5]$$

$$0 \;=\; r_{(i),j}(\mathbf{x}(t_i^c - 0), \mathbf{x}(t_i^c + 0), \omega, t_i^c), \;\; j = 1, \ldots, n_{r_{(i)}}, \; i = 1, \ldots, n_c, \qquad [6]$$

$$x_j(0) \;=\; x_{0,j}, \quad x_k(t_f) \;=\; x_{f,k}, \quad x_l(t_i^c + 0) \;=\; R_{(i),l}(\mathbf{x}(t_i^c - 0), \omega, t_i^c), \qquad [7]$$

where j, k, l are elements from subsets of $\{1, 2, \ldots, n_x\}$ and $x_{0,j}, x_{f,k}$ are given real constants. For a convenient notation of the objective [2] and the switching conditions [6] the notation $t_{n_c+1}^c + 0 = t_0^c = 0$ has been used with $\mathbf{x}(t \pm 0) := \lim_{\delta \to 0} \mathbf{x}(t \pm \delta)$, $\delta > 0$. Furthermore, linear constraints are imposed on the binary control vector

$$\mathbf{l}_{\min} \le \mathbf{A}\omega \le \mathbf{l}_{\max}, \;\; \mathbf{A} \in \mathbb{R}^{n_A \times n_\omega}, \;\; \mathbf{l}_{\min}, \mathbf{l}_{\max} \in \mathbb{R}^{n_A}. \qquad [8]$$

The solutions of the MIOCP are the optimal (open loop) trajectories of $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, $0 \le t \le t_f$, the optimal phase transition times t_i^{c*}, the possibly free final time t_f^* and the optimal binary parameter vector ω^*. The switchings at the event times t_i^c may either be induced externally as a result of *controlled switching* as in the motorized traveling salesmen problems (see Sect. 3) or as in the optimal hybrid control of an underactuated 2-DOF manipulator [HAR 00b]. The switchings may also be induced internally which is called *autonomous switching* which occurs, e. g., when searching for the minimum energy gait patterns and trajectories of a quadruped walking machine [HAR 00a].

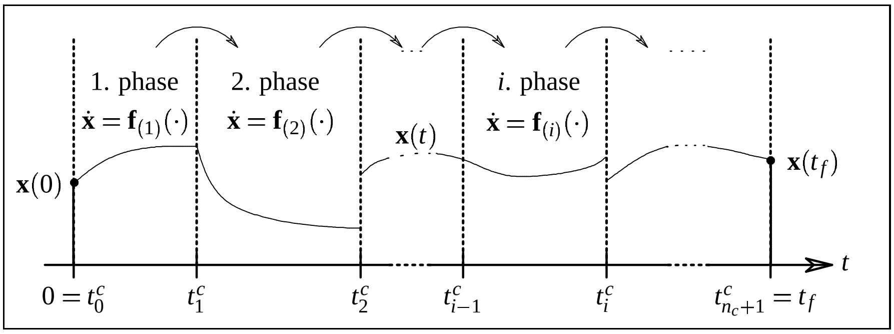

Figure 1. *Continuous state dynamics defined in multiple phases, phase transitions occur at switching times* t_i^c

REMARK 1. A general class of hybrid optimal control problems is defined by determing the optimal hybrid — i. e., continuous $\mathbf{u}$ and discrete $\mathbf{v} : [0, t_f] \to \mathcal{V} \subset \mathbb{Z}^{n_v}$ — control trajectories such that a hybrid cost index $\int_0^{t_f} L(\mathbf{x}, \mathbf{u}, \mathbf{q}, \mathbf{v})\, \mathrm{d}t$ is minimized subject to the system dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{q}, \mathbf{v}, t)$ and further constraints, where $\mathbf{x}$ denotes the continuous state and $\mathbf{q} : [0, t_f] \to \mathcal{Q} \subset \mathbb{Z}^{n_q}$ the discrete state [BUS 00, STR 00]. Assuming a finite, either given or bounded, number of switchings for both $\mathbf{q}$ and $\mathbf{v}$, then the hybrid optimal control problem can be transformed into a MIOCP with integer variables which may be represented by binary variables [BUS 00, STR 00]. Thus, the "nature" of the binary control vector ω appearing in the MIOCP is ambivalent. On the one hand it represents the discrete control variable $\mathbf{v}$ that controls the order and types of phase transitions, on the other hand it also represents the discrete state $\mathbf{q}$ in each phase.

2. Decomposition using branch and bound and sparse direct collocation methods

No general solution techniques which address the class of MIOCP's are currently available. A framework for modeling and (optimally) controlling mixed logical dynamical systems described by linear dynamic equations subject to linear inequalities involving real and integer variables has been proposed by [BEM 99]. The on-line optimization problems resulting from a predictive control scheme are solved numerically by application of a mixed-integer quadratic programming branch-and-bound method. However, the approach is not applicable to our class of MIOCP's with *nonlinear* dynamics equations subject to *nonlinear* constraints.

By discretizing the continuous state and control variables of MIOCP's, the resulting mixed-integer nonlinearly constrained optimization problems (MINLPs) are in general nonconvex. Consequently, the recently developed numerical methods for convex MINLPs [ADJ 99, GRO 97] cannot be applied since the bounding properties of the relaxed problem cannot be achieved [ALL 97]. However, there is another fundamental reason why MINLP techniques are not suited for general MIOCP's: the value of the discrete variable determines the sequence, type and number of phase dynamics. Thus, the dynamics in a phase and even the dimension or number of constraints may be completely different for different values of the discrete variable. A discretization of the continuous variable, however, must depend on the dynamics and constraints to be any effective or defined at all.

A prerequisite for the numerical solution of MIOCP's is the ability to efficiently solve optimal control problems with nonlinear dynamical equations defined in multiple phases subject to nonlinear constraints. The recently developed sparse direct collocation method DIRCOL (Sect. 2.2) satisfies these requirements.

A naive solution approach to MIOCP is to enumerate the feasible discrete control space, i. e., determining all $\omega \subset \{0,1\}^{n_\omega}$ satisfying Eq. [8], and solving all the resulting "continuous" multi-phase optimal control problems related to each of the binary control values. However, even for moderate dimensions of ω, this approach is not feasible because of the combinatorial complexity of the discrete part of the hybrid dynamic optimization problem (cf. example of Sect. 3.1).

2.1. *Decomposition using branch and bound*

The motivation for a decomposition approach is to potentially avoid an explicit enumeration of the entire feasible discrete control space $\subset \{0,1\}^{n_\omega}$ by solving sequences of problems providing rigorous upper (non increasing) and lower (non decreasing) bounds on the MIOCP performance index that converge in a significantly smaller number of iterations.

To obtain an upper bound, the components of ω are set to a combination of 0 or 1 satisfying the linear constraints [8]. Hereby, the MIOCP is reduced to a "continuous", multi-phase optimal control problem (primal problem), whose solution is assumed and its global optimum yields a rigorous upper bound on the MIOCP performance index. A lower bound is obtained by the global optimal solution to a multi-phase optimal control problem for a binary control vector with partially relaxed components $0 \leq \omega_i \leq 1$, $i \in \{1,2,\ldots,n_\omega\}$.

A branch and bound technique may start from the root of the binary search tree, where all binary variables are relaxed if the underlying MIOCP allows it. In each inner node some of the binary variables are 0 or 1, all others are relaxed. At the ends of the search tree, all binary variables are 0 or 1. At an inner node, the global solution of the corresponding multi-phase optimal control problem yields a lower bound on the achievable performance index for all nodes of the subtree. If the lower bound in a node is greater than the current best upper bound of the whole search tree, then all subsequent branches from this node can be cut off. Thus, the efficiency of the B&B binary tree search strongly depends on good lower and upper bounds, especially on a good initial upper bound produced by a good initial estimate of ω^*. Various and general strategies for B&B exist on how to select the branching variable (such as first free variable or maximum fractional part) and how to explore the tree (such as depth first or breadth first). Currently, it is not clear yet which strategy performs best for which type of MIOCP.

REMARK 2. B&B for the binary control vector requires existence of solutions to relaxed MIOCP's, or more precisely, the existence of continuous relaxations to the MIOCP. For some MIOCP's, relaxations may exist naturally, although the relaxed problem does not need to be of any physical significance with respect to the underlying application. Usually additional modeling effort will be required in defining suitable "meta"-MIOCP's allowing useful relaxations analogously to the definition of super-structures for mixed-integer nonlinear programming problems [ADJ 99].

REMARK 3. As it is to be expected that some modeling effort for the MIOCP must be made before applying any numerical methods, it has been suggested to derive suitably simplified and problem specific "screening models" [ALL 97]. A screening model can be solved to simultaneously guarantee global optimality and to yield a rigorous lower bound on the solution to the MIOCP, thus avoiding the need for dealing with relaxed MIOCP's. An application for a simple batch process development has successfully been investigated in [ALL 97].

REMARK 4. The B&B binary tree search incorporates a high parallelism that can be employed in future investigations on a massively parallel computing platform as the communication between processors can be neglected when compared to the computational time for a multi-phase optimal control problem in a node [GOU 00].

REMARK 5. The linear constraints [8] on the binary control vector are used during the B&B approach to eliminate nodes before actually solving any relaxed MIOCP's.

REMARK 6. The challenge in solving relaxed MIOCP's during the binary tree search cannot be underestimated. There is no numerical method available that solves optimal control problems with nonlinear dynamics defined in multiple phases and subject to nonlinear constraints and with phase transitions at unknown times *guaranteeing* the global optimum or that even guarantees a locally optimal solution in general at all. However, not only is the global optimum of interest. For many types of MIOCP's, even a "good" solution obtained by the proposed approach that significantly improves the initial guess will be highly appreciated.

2.2. *Sparse direct collocation for multi-phase optimal control problems*

The "continuous" multi-phase optimal control problems that are obtained from Eqs. [1] – [8] by having some components of ω fixed to 0 or 1 and some relaxed still include several discrete event effects. For example, the order of phase transitions is given but not the times of switchings. Further switching points may describe the times when a state or control variable inequality constraint [3] becomes active or inactive during a phase.

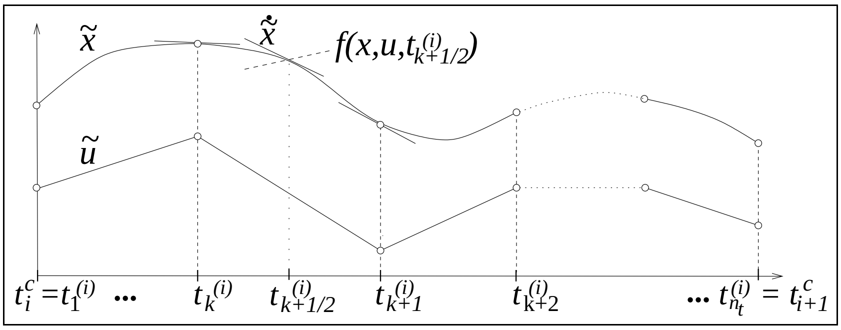

Figure 2. *Direct collocation parameterization of continuous state/control variables*

Numerical optimal control methods based on the Euler-Lagrange differential equations (EL-DEQs) and the Maximum Principle (MP) can be mainly divided into two classes: direct and indirect methods [STR 92]. Indirect methods approximate a solution by explicitly solving first and second order optimality conditions resulting from EL-DEQs and the MP. For reasons already discussed in [BUS 00, STR 92, STR 00] they are not flexible enough for the purpose needed here. Direct methods are based on a transcription of optimal control problems into (finite dimensional) nonlinearly constrained optimization problems (NLPs) either by direct shooting or direct collocation [BET 98, STR 92]. Direct methods promise high flexibility and robustness when solving optimal control problems numerically to low or moderate accuracies.

In many practical applications the problem functions have only low, local differentiability properties, i. e., discontinuities in the first or second order derivatives. Thus, obtaining a useful gradient approximation for shooting-type discretizations is much more involved since a numerical sensitivity analysis of initial value problems with switching points must be carried out, see e. g., [ALL 97]. On the other hand, for a collocation-type discretization, only a careful, but much cheaper finite difference approximation is sufficient with usually no need for special treatment of discontinuities in first or second derivatives by switching functions. Additionally appealing in the direct collocation approach is the potentially faster computation compared to direct shooting because the ODE simulation [1] and the control optimization problems [2], [3] are solved simultaneously for collocation and not iteratively as for shooting. To achieve full speed-up for collocation, the NLP sparsity must be fully exploited [BAR 98, BET 98, STR 00]. Otherwise the NLP size will severly limit the efficiency.

We apply a discretization of $\mathbf{x}$ by piecewise cubic Hermite polynomials $\tilde{\mathbf{x}}(t) = \sum_j \alpha_j \hat{\mathbf{x}}_j(t)$ and of $\mathbf{u}$ by piecewise linear functions $\tilde{\mathbf{u}}(t) = \sum_k \beta_k \hat{\mathbf{u}}_k(t)$ on a discretization grid $t_i^c = t_1^{(i)} < t_2^{(i)} < \ldots < t_{n_t^{(i)}}^{(i)} = t_{i+1}^c$ in each phase (Fig. 2) [HAR 87, STR 93]. The equations of motions [1] are pointwise fulfilled at the grid points and at their respective midpoints resulting in a set of nonlinear NLP equality constraints $\mathbf{a}(\mathbf{y}) = 0$ (collocation at Lobatto points). Any control or state variable inequality constraints are to be satisfied at the grid points resulting in a set of nonlinear NLP inequality constraints $\mathbf{b}(\mathbf{y}) \geq 0$. Here, $\mathbf{y}$ denotes the n_y parameters of the parameterization

$$\mathbf{y} = (\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ..., \mathbf{p}, t_1^c, \ldots, t_{n_c}^c, t_f)^T.$$

where $p_i \in [0,1], i = 1, \ldots, n_p$, denotes the relaxed binary variables. The resulting nonlinearly constrained optimization problem reads as

$$\text{NLP:} \quad \min_{\mathbf{y}} \phi(\mathbf{y}) \quad \text{subject to} \quad \mathbf{a}(\mathbf{y}) = 0,\ \mathbf{b}(\mathbf{y}) \geq 0,$$

where ϕ denotes the parameterized cost index [2].

A carefully selected discretization $\tilde{\mathbf{u}}$, $\tilde{\mathbf{x}}$ must satisfy certain convergence properties. One requirement is that the discretized solution must approximate a solution of the EL-DEQs and the Maximum Principle if the grid becomes fine enough, i. e., for $n_t^{(i)} \to \infty$ and $\max\{t_{k+1}^{(i)} - t_k^{(i)} : k = 1, \ldots, n_t^{(i)} - 1\} \to 0$. This property is satisfied by the direct collocation method [STR 93, STR 00]. Another great advantage of this approach is that it provides reliable estimates $\tilde{\lambda}$ of the adjoint variable trajectory along the discretization grid. These estimates are derived from the Lagrange multipliers of the NLP [STR 93, STR 00]. They enable a verification of optimality conditions of the discretized solution although the EL-DEQs have not been solved explicitly. Local optimality error estimates can be derived that enable efficient strategies for successively refining a first solution on a coarse grid [STR 99b, STR 00]. Thus, a sequence of related NLPs must be solved whose dimensions increase with the number of grid points.

NLPs can be solved most efficiently numerically by SQP methods. In each SQP iteration a current guess of the solution $\mathbf{y}^*$ is improved by the solution of a quadratic subproblem derived from a quadratic approximation of the Lagrangian of the NLP subject to the linearized constraints [BAR 98, GIL 97].

The NLPs resulting from a direct collocation discretization have several special properties [STR 00]: The NLPs are of large-scale with very many variables and very many constraints. Most of the NLP constraints are active at the solution, e. g., the equality constraints from collocation. Thus, the number of "free" NLP variables is much smaller than the total number of variables n_y. The Jacobians $(\nabla\mathbf{a}(\mathbf{y}), \nabla\mathbf{b}(\mathbf{y}))$ of the NLP are sparse and structured. Only a few percent of the elements will be nonzero, and the percentage decreases as the number of grid points increases. The NLP objective $\phi(\mathbf{y})$ only depends on a few, fixed number of variables independently of the actual grid size when the objective [2] is of Mayer type, i. e., $L \equiv 0$. All these features can be utilized by the recently developed software DIRCOL [STR 99b] which uses the recent large-scale SQP method SNOPT [GIL 97]. The Hessian of the NLP Lagrangian is approximated by limited-memory quasi-Newton updates and a reduced Hessian algorithm is used for solving the QP subproblems. The null-space matrix of the working set in each iteration is obtained from a sparse LU factorization.

The computational speedup achievable by fully exploiting the NLP structure is more than a factor of one hundred for typical discretized optimal control problems when compared to standard "dense" SQP methods [STR 00].

REMARK 7. DIRCOL [STR 99b] is especially suited for solving the relaxed MIOCP's because of its exceptional robustness and efficiency. Typically only a crude initial guess of parts or all of the solution trajectories of a relaxed MIOCP can be provided. This holds especially for the phase transition times which may have quite different positions at the solution than as they can be provided initially. The large movements of events from their initial to their final position during the course of the optimization method usually pose high difficulties for other methods. On the other hand, an initially crude solution estimate on a rather coarse discretization grid is not a handicap to DIRCOL but the usual way how the solution procedure begins. Finally, a relaxed MIOCP does not need to be solved to the finest grid needed for a given tolerance. If the optimality error estimate of the performance index provided by DIRCOL is taken into account, then the computational effort for solving for a refined grid can be avoided if the error estimate indicates that the result for the current grid cannot be below the current upper bound.

3. Hybrid optimal control problems for traveling salesmen

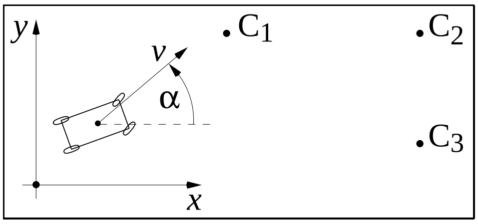

Figure 3. *Nomenclature for the motorized traveling salesman problem*

In this section we introduce and discuss two new benchmark problems for hybrid optimal control. Their major benefits are that they intuitively help understanding the basic problems encountered in hybrid, mixed-integer optimal control, and that they are easily scalable in terms of their combinatorial complexity, i. e., the number of cities involved, as well as their continuous complexity, i. e., the vehicle dynamics model. The presented approach has also been applied for optimal hybrid control of an underactuated 2-DOF manipulator [HAR 00b] and for minimum energy gait patterns and trajectories of a quadruped walking machine [HAR 00a].

3.1. *The motorized traveling salesman problem*

The Traveling Salesman Problem (TSP) is one of the most prominent members of combinatorial optimization problems [COO 98, LAW 85]. We introduce a hybrid dynamical extension of the TSP to demonstrate the strong interaction of continuous and discrete dynamics in hybrid optimal control [STR 00]. Here, the salesman is supposed to drive a car (Fig. 3). This problem was presented originally in [STR 99a].

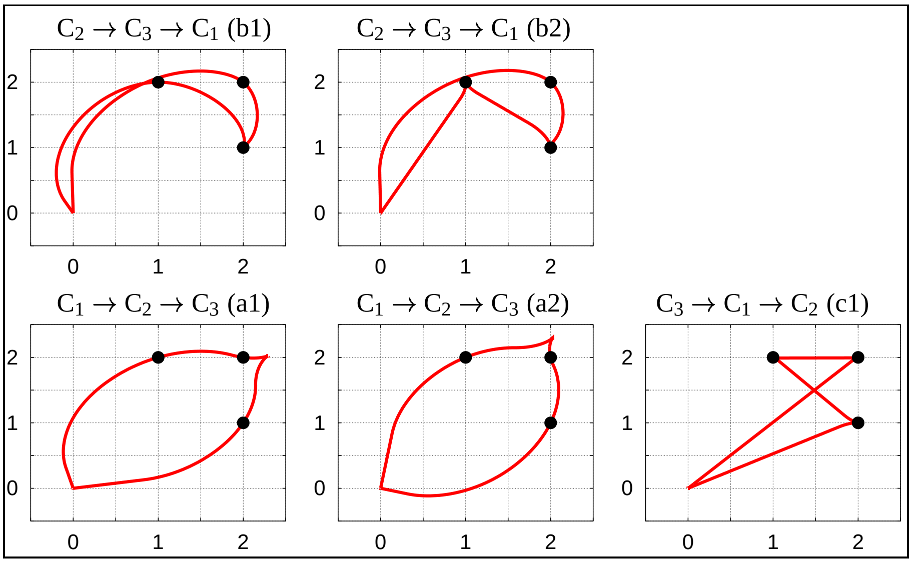

Figure 4. *The five minimum time solution candidates obtained for the motorized traveling salesman problem with three cities*

The problem is described as follows: *A salesman spends his time visiting n_c cities cyclically. In one tour he passes by each city just once and finishes up in the origin where he started. In what order should he visit them to minimize the overall travel time?* The task is to determine the steering angle velocity γ and the accelerating or braking force β (two continuous controls) and the order (discrete control) in which the n_c cities $C_k = (x_k^c, y_k^c)^T$ have to be visited such that the overall travel time is minimized. There are no further restrictions on the path, i. e., the "road", in the (x, y)-plane.

A simplified kinematical model of the car is given by

$$\begin{array}{rcllrl} \dot{x}(t) & = & v(t)\cos(\alpha(t)), & x(0) = 0 = x(t_f), & & \\ \dot{y}(t) & = & v(t)\sin(\alpha(t)), & y(0) = 0 = y(t_f), & & \\ \dot{v}(t) & = & \beta(t), & v(0) = 0 = v(t_f), & |\beta(t)| & \leq 1, \\ \dot{\alpha}(t) & = & \gamma(t), & \alpha(0),\ \alpha(t_f)\ \text{free}, & |\gamma(t)| & \leq 1. \end{array} \qquad [9]$$

A phase describes the trip between two cities. Thus, the number of phases is $n_c + 1$. Let t_i^c denote the time when the i-th city is passed: $0 = t_0^c < t_1^c < \ldots < t_{n_c}^c < t_{n_c+1}^c = t_f$. Then the motorized TSP (MTSP) is formulated as a MIOCP according to Sect. 1 with $\mathbf{u} = (\gamma, \beta)^T$, $\mathbf{x} = (x, y, v, \alpha)^T$, $\omega \in \{0,1\}^{n_c \times n_c 1}$, and

$$\min_{\mathbf{u},\,\omega} J[\mathbf{u}, \omega] := t_f \qquad [10]$$

[1]For a convenient notation of the MTSP, ω is here defined as a $n_c \times n_c$-matrix instead of a $n_\omega = n_c^2$-dimensional vector as in the general MIOCP setting of Sect. 1.

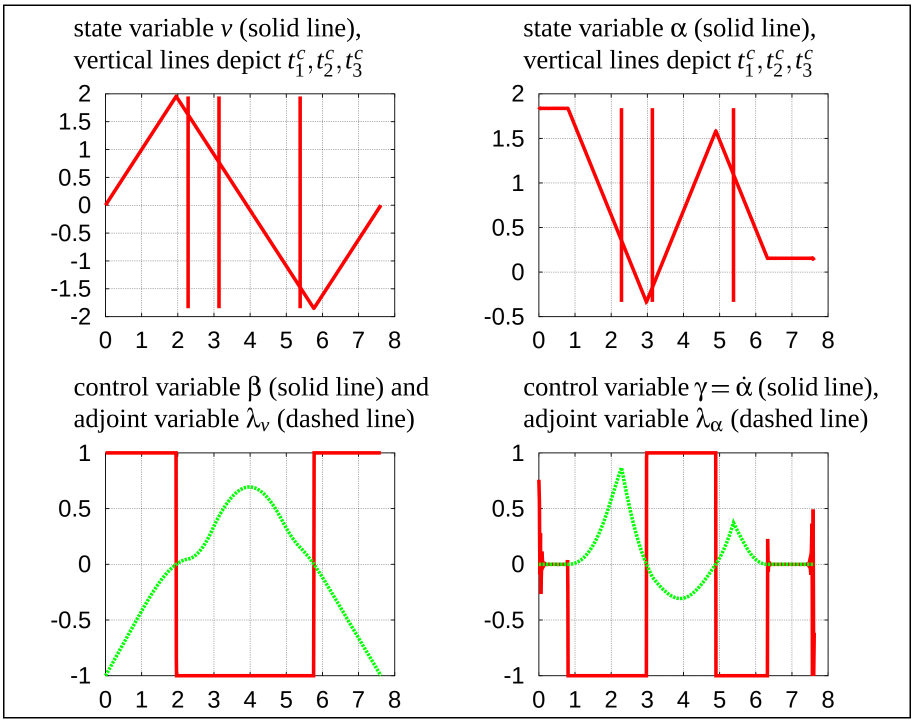

Figure 5. *Solution $C_1 \to C_2 \to C_3$ (a1) for the motorized traveling salesman problem*

$$r_{(i)}(\mathbf{x}(t_i^c - 0), \mathbf{x}(t_i^c + 0), \omega, t_i^c) := \begin{pmatrix} x(t_i^c - 0) \\ y(t_i^c - 0) \end{pmatrix} - \sum_{k=1}^{n_c} \omega_{i,k} \begin{pmatrix} x_k^c \\ y_k^c \end{pmatrix} \qquad [11]$$

$$\mathbf{x}(t_i^c + 0) = R_{(i)}(\mathbf{x}(t_i^c - 0), \omega, t_i^c - 0) := \mathbf{x}(t_i^c - 0) \qquad [12]$$

$$\sum_{i=1}^{n_c} \omega_{i,k} = 1, \quad \sum_{k=1}^{n_c} \omega_{i,k} = 1, \quad 0 \leq \omega_{i,k} \leq 1. \qquad [13]$$

The constraints [11] and [13] ensure that each city is visited exactly once on each tour. Each tour, i. e., each feasible ω is a permutation of n_c cities. The problem is autonomous and without active state constraints. Thus, the MTSP is symmetric as a tour driven forward or backwards yields the same travel time. Therefore, the number of possible tours is $(n_c)!/2$. The number of tours does not increase polynomially with the number of cities. For example, for 3 cities the number of tours is 3, for 5 cities it is 60, for 10 cities it is 1 814 400, and for 50 cities it is approximately $\approx 1.52 \times 10^{64}$. Now if we assume that all minimum time tours for 5 cities can be computed in one second inclusive the selection of the best one, then to solve the problem for 20 cities in this way by total enumeration will need approximately $20!/5! \approx 2.03 \times 10^{16}\,\mathrm{s} \approx 643$ million years. The TSP is NP-complete. Therefore, total enumeration of the feasible discrete variable space is not a useful approach.

tour / solution	t_f	t_1^c	t_2^c	t_3^c
1-2-3 (a1)	7.6166	2.286	3.139	5.383
1-2-3 (a2)	7.6166	2.232	4.481	5.332
2-3-1 (b1)	10.0204	3.510	5.302	7.312
2-3-1 (b2)	10.7394	3.570	5.336	7.787
3-1-2 (c1)	10.7644	2.945	5.331	7.352

Table 1. *Total travel times and corresponding switching times for the solution candidates of the motorized traveling salesman problem with three cities*

For $n_c = 3$ cities $C_1 = (1,2)$, $C_2 = (2,2)$, $C_3 = (2,1)$, the three possible sequences ("tours") are

$$\begin{array}{lll} (a) & C_1 \to C_2 \to C_3 & (\text{same as } C_3 \to C_2 \to C_1) \\ (b) & C_2 \to C_3 \to C_1 & (\text{same as } C_1 \to C_3 \to C_2) \\ (c) & C_3 \to C_1 \to C_2 & (\text{same as } C_2 \to C_1 \to C_3)\,. \end{array}$$

For each of the tours the continuous controls and switching times have been optimized using the direct collocation method of Sect. 2.2 with respect to the terminal time t_f for a given discrete variable, i. e., order of cities, i. e., sequence of phases. To start the iterative direct collocation method, initial guesses for the switching points consisting of $t^c_{i,\text{estimate}} = i$, $i = 1,\ldots,n_c+1$, are used. A linear interpolation of the coordinates of the cities is applied as an initial guess for x and y, whereas v, α, β, and γ were initially set to zero.

There are $2^{n_c \cdot n_c} = 512$ possible combinations for ω and the binary search tree of Sect. 2.1 consists of $2^{n_c \cdot n_c + 1} - 1 = 1023$ elements, but only $(n_c)! = 6$ are feasible with respect to Eq. [13]. Because of the symmetry only three candidates for the discrete variable remain.

The five solution trajectories obtained for the three discrete candidates by solving the corresponding multi-phase optimal control problems numerically for the described and for perturbed initial guesses are depicted in Fig. 4. The obtained data for the final time and switching points of each solution are given in Table 1.

The problem has several local minima. Even the best solution found for the sequence $C_1 \to C_2 \to C_3$ is not unique. The two solutions (a1) and (a2) are equivalent because of the symmetry of the problem for this sequence and yield the same t_f^*. Also for the sequence $C_2 \to C_3 \to C_1$ two local minima (b1) and (b2) have been found with different values for t_f (Fig. 4). Even more may exist for the current problem. It must be expected that the number of local minima increases with the number of cities investigated. The many local minima are a consequence of the problem setting and not of the numerical method applied. Thus, it can not be expected that second order necessary conditions may help to reduce the number of solution candidates significantly. As second order sufficiency conditions cannot be checked easily numerically, the only way left is to solve the problem for several different initial guesses and then to select the best local minima.

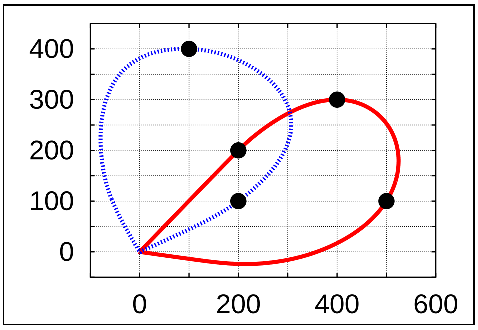

Figure 6. *The optimal tours of two cooperating motorized salesmen to five cities*

The estimates of the adjoint variables provided by the direct collocation method are used to check the computed solutions for consistency with the EL-DEQs and the Maximum Principle. Both controls appear linearly in the Hamiltonian $\mathcal{H} := \lambda_x v \cos\alpha + \lambda_y v \sin\alpha + \lambda_v \beta + \lambda_\alpha \gamma$. Thus, $\lambda_v > 0 \Rightarrow \beta = -1$ and $\lambda_v < 0 \Rightarrow \beta = +1$ are conditions satisfied by the computed solution (Fig. 5). An analogous argument holds for γ and λ_α (Fig. 5). In addition, two intervals $[0, 0.8]$, $[6.33, t_f]$ appear where $\lambda_\alpha(t) = 0$ and, thus, γ is a singular control of order 1. The existence of these two singular control intervals is also indicated by another argument: Because $\alpha(0)$ and $\alpha(t_f)$ are free, $\lambda_\alpha(0) = \partial\varphi/\partial\alpha(0) = 0 = \partial\varphi/\partial\alpha(t_f) = \lambda_\alpha(t_f)$ must hold.

3.2. *A team of two cooperating motorized traveling salesmen*

We consider a hybrid, cooperative dynamic game extension of the MTSP for two salesmen representing the class of rather autonomous dynamic agents that cooperate optimally [GLO 00, STR 99a, STR 00]: *Two motorized salesmen start their tours in the origin and return to it after each city has been visited just once by one and only one of them. In what order should the cities be visited by each of the salesmen to minimize the overall travel time?*

Under the assumption of Pareto-optimality of the cooperating controls, candidates for Pareto-optimal solutions can be determined by optimal control methods [LEI 74]. Let $\mathbf{x}_\mathrm{I} = (x_\mathrm{I}, y_\mathrm{I}, v_\mathrm{I}, \alpha_\mathrm{I})^T$ and $\mathbf{x}_\mathrm{II} = (x_\mathrm{II}, y_\mathrm{II}, v_\mathrm{II}, \alpha_\mathrm{II})^T$ denote the continuous state variables of the first and second salesman, respectively, and $\mathbf{u}_\mathrm{I} = (\gamma_\mathrm{I}, \beta_\mathrm{I})^T$ and $\mathbf{u}_\mathrm{II} = (\gamma_\mathrm{II}, \beta_\mathrm{II})^T$ denote their continuous control variables. Then the equations of motion for both salesmen are described by a system of differential equations [9]. With the binary control vector $\omega \in \{0,1\}^{(n_c+1)\times n_c}$ the cooperative, hybrid dynamic game of the two motorized salesmen can be stated as a mixed-binary optimal control problem

$$\min_{\mathbf{u}_I, \mathbf{u}_{II}, \omega} J[\mathbf{u}_I, \mathbf{u}_{II}, \omega] \quad := \quad t_f \qquad [14]$$

$$\text{subject to} \quad \dot{\mathbf{x}}_\mathrm{I} \;=\; \mathbf{f}(\mathbf{x}_\mathrm{I}, \mathbf{u}_\mathrm{I}), \quad x_{\mathrm{I},i}(0) = 0 = x_{\mathrm{I},i}(t_f), \quad i = 1,2,3, \qquad [15]$$

$$\dot{\mathbf{x}}_\mathrm{II} \;=\; \mathbf{f}(\mathbf{x}_\mathrm{II}, \mathbf{u}_\mathrm{II}), \quad x_{\mathrm{II},j}(0) = 0 = x_{\mathrm{II},j}(t_f), \quad j = 1,2,3, \qquad [16]$$

$$
\begin{aligned}
\mathbf{r}_{(i)} &= \mathbf{r}_{(i)}(\mathbf{x}_{\rm I}(t_i^c-0),\mathbf{x}_{\rm II}(t_i^c-0),\mathbf{x}_{\rm I}(t_i^c+0),\mathbf{x}_{\rm II}(t_i^c+0),\omega,t_i^c) \\
&= \omega_{n_c+1,i}\begin{pmatrix} x_{\rm I}(t_i^c-0) \\ y_{\rm I}(t_i^c-0)\end{pmatrix} + (1-\omega_{n_c+1,i})\begin{pmatrix} x_{\rm II}(t_i^c-0) \\ y_{\rm II}(t_i^c-0)\end{pmatrix} \\
&\quad -\sum_{k=1}^{n_c}\omega_{i,k}\begin{pmatrix} x_k^c \\ y_k^c \end{pmatrix}, \qquad i=1,\ldots,n_c, && [17] \\
\mathbf{x}_{\rm I}(t_i^c+0) &= \mathbf{R}_{(i),\rm I}(\mathbf{x}_{\rm I}(t_i^c-0),\omega,t_i^c-0) := \mathbf{x}_{\rm I}(t_i^c-0), && [18] \\
\mathbf{x}_{\rm II}(t_i^c+0) &= \mathbf{R}_{(i),\rm II}(\mathbf{x}_{\rm II}(t_i^c-0),\omega,t_i^c-0) := \mathbf{x}_{\rm II}(t_i^c-0), i=1, && [19] \\
& \quad i=1,\ldots,n_c, \\
|v_{\rm I}(t)| &\le v_{\max}, \qquad |v_{\rm II}(t)| \le v_{\max}, && [20] \\
|u_{{\rm I},j}(t)| &\le u_{j,\max}, \qquad |u_{{\rm II},j}(t)| \le u_{j,\max}, \qquad j=1,2, && [21] \\
& \sum_{i=1}^{n_c}\omega_{i,k}=1, \qquad \sum_{k=1}^{n_c}\omega_{i,k}=1. && [22]
\end{aligned}
$$

The conditions [17] ensure that at the switching point t_i^c one of the cities C_k is visited by one of the salesmen. The conditions [18] and [19] describe that the state variables are continuous at a switching point t_i^c. The linear constraints [22] guarantee that each city is visited exactly once on each tour by (only) one of the salesmen. If a relaxation of the problem is investigated with some or all $\omega_{i,k} \in \mathbb{R}$, then

$$0 \le \omega_{i,k} \le 1, \quad i=1,\ldots,n_c+1, \;\; k=1,\ldots,n_c, \qquad [23]$$

must hold in addition.

The solution for a problem with 5 cities $C_1=(100,400)$, $C_2=(200,200)$, $C_3=(500,100)$, $C_4=(400,300)$, $C_5=(200,100)$, and $u_{2,\max}=\sqrt{7}$ and $v_{\max}=\sqrt{1000}$ is displayed in Fig. 6. There exist $2\sum_{i=1}^{n_c} i! = 306$ possible tours. A problem formulation with 36 binary variables [GLO 00] results in a binary search tree with $2^{37}-1 = 1.374...\times 10^{11}$ nodes, but only 306 combinations being feasible with respect to the corresponding linear constraints [8], i. e., eqs. [22] and [23]. The computation for B&B with a depth-first strategy takes 2 hours and 45 minutes determining the solution for 151 nodes on a Linux-PC with a Pentium III/500 MHz processor. The computational time for solving a single, relaxed MIOCP varied between a few seconds and three minutes, while the average computational time per problem was about one minute.

REMARK 8. The minimum time objective only determines the solution for one salesman uniquely, and t_f^* only gives an upper bound for the travel time of the second salesman. If the second salesman is additionally required to minimize a further criterion such as the energy subject to an overall tour time less than or equal to t_f^*, then the trajectories of the second salesman will also be (locally) uniquely determined. For example, if the objective [14] is replaced by

$$\min_{\mathbf{u}_I,\mathbf{u}_{II},\omega} J[\mathbf{u}_I,\mathbf{u}_{II},\omega] := t_f + \rho\int_0^{t_f}\sum_{i=1}^{n_u}\left(u_{{\rm I},i}^2(t)+u_{{\rm II},i}^2(t)\right)\mathrm{d}t \qquad [24]$$

with a small weight $\rho > 0$, then the solution will be locally uniquely determined again.

4. Conclusions

A first method for solving numerically general hybrid optimal controls has been presented. A decomposition approach is applied to efficiently solving fairly general MIOCP's by branch-and-bound and sparse direct collocation. The decomposition relies on the capabilities of the recently developed, sparse direct collocation method DIRCOL for efficiently solving optimal control problems with multivariable nonlinear dynamics defined in multiple phases subject to nonlinear constraints. Furthermore, two new benchmark hybrid optimal control problems are presented for a single and two cooperating motorized traveling salesmen.

Focus of further research will be the derivation of better lower bounds, e. g., from a dualization of a hybrid maximum principle, and, thus, a more efficient binary search. This may also help in extending the sparse direct collocation method to attain global solutions in case of multiple local minima. Further methods are needed to determine good initial estimates of the discrete variables yielding good upper bounds and possibly reducing the need for a global binary search. Finally, in working towards synthesizing feedback controllers for MIOCP's the development of theories for perturbation and sensitivity analysis of such problems are needed. Despite some encouraging first results, research in nonlinear hybrid optimal control is still at its beginning.

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