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Decomposition of Mixed-Integer Optimal Control Problems Using Branch and Bound and Sparse Direct Collocation

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Abstract

A large class of optimal control problems for hybrid dynamic systems can be formulated as mixed-integer optimal control problems (MIOCPs). It is the intrinsic combinatorial complexity, in addition to the nonlinearity of the continuous, multi-phase optimal control problems that is largely responsible for the challenges in the theoretical and numerical solution of MIOCPs. We present a new decomposition approach to numerically solving fairly general MICOPs with binary control variables. A Branch and Bound (B&B) technique is applied to efficiently search the entire discrete solution space performing a truncated binary tree search for the discrete variables maintaining upper and lower bounds on the performance index. The partially relaxed binary variables at an inner node define an optimal control problem with dynamic equations defined in multiple phases. Its global solution provides a lower bound on the performance index for all nodes of the subtree. If the lower bound for a given subtree is greater than the current global upper bound then that entire subtree need no longer be searched. The many optimal control problems with nonlinear, continuous state dynamics defined in multiple phases subject to nonlinear constraints are solved most efficiently by a sparse direct collocation transcription. Hereby, the multi-phase optimal control problem is transcribed to a sparse, large-scale nonlinear programming problem being solved efficiently by a tailored SQP method. Despite the high efficiency of the sparse direct collocation method, the efficiency of the decomposition technique for MIOCPs strongly depends on the determination of good lower and upper bounds on the performance index being used to fathoming entire subtrees throughout the binary tree search. The proposed approach is successfully applied to two new benchmark problems for hybrid optimal control: a motorized traveling salesman and a team of two cooperating, motorized salesmen.

Keywords: Nonlinear hybrid dynamical systems, multi-phase dynamical systems, hybrid optimal control, mixed-integer optimal control, branch-and-bound, sparse direct collocation, motorized traveling salesman problem

1 Mixed-Integer Optimal Control

We consider a nonlinear dynamical system defined in n_c+1 phases $[t_i^c, t_{i+1}^c]$, $i = 0,\dots,n_c$, (cf. Fig. 1)

$$\dot{\mathbf{x}}(t) \;=\; \mathbf{f}_{(i)}(\mathbf{x}(t),\mathbf{u}(t),\omega,t), \;\; t_i^c \le t < t_{i+1}^c, \qquad (1)$$

with a piecewise continuously differentiable state variable $\mathbf{x} : [0,t_f] \to \mathbb{R}^{n_x}$, a piecewise continuous control variable $\mathbf{u} : [0,t_f] \to \mathbb{R}^{n_u}$, and a binary control vector $\omega \in \{0,1\}^{n_\omega}$. The transition from phase i to phase $i+1$ takes place at the usually unknown event t_i^c, $i = 1,\dots,n_c$, i.e. a switching point. The mixed-integer optimal control problem (MIOCP) is defined as minimizing the real-valued, hybrid performance index

$$\begin{aligned} J[\mathbf{u},\omega] \;=\; & \sum_{i=1}^{n_c+1} \varphi_{(i)}\left(\mathbf{x}(t_i^c-0),\mathbf{x}(t_i^c+0),\omega,t_i^c\right) \\ & + \sum_{i=0}^{n_c} \int_{t_i^c}^{t_{i+1}^c} L_{(i)}(\mathbf{x}(t),\mathbf{u}(t),\omega,t)\,\mathrm{d}t \qquad (2) \end{aligned}$$

with respect to the continuous control variable $\mathbf{u}$ and the discrete control vector ω and subject to state and control variable inequality and equality constraints

$$0 \;\le\; g_{(i),j}(\mathbf{x}(t),\mathbf{u}(t),\omega,t), \quad j = 1,\dots,n_{g(i)}, \quad (3)$$

$$0 \;=\; h_{(i),j}(\mathbf{x}(t),\mathbf{u}(t),\omega,t), \quad j = 1,\dots,n_{h(i)}. \quad (4)$$

Number and type of the constraints differ from one phase to the other. Furthermore, at initial, final, and switching times implicit switching conditions may hold

$$0 \;=\; r_{(0),j}(\mathbf{x}(0),\mathbf{x}(t_f),\omega,t_f), \;\; j = 1,\dots,n_{r_{(0)}}, \quad (5)$$

$$0 \;=\; r_{(i),j}(\mathbf{x}(t_i^c-0),\mathbf{x}(t_i^c+0),\omega,t_i^c), \qquad (6)$$

$j = 1,\dots,n_{r_{(i)}}$, $i = 1,\dots,n_c$, as well as explicit conditions

$$\begin{aligned} x_j(0) &= x_{0,j}, \quad x_k(t_f) = x_{f,k}, \\ x_l(t_i^c+0) &= R_{(i),l}(\mathbf{x}(t_i^c-0),\omega,t_i^c), \end{aligned} \qquad (7)$$

where j,k,l are elements from subsets of $\{1,2,\dots,n_x\}$ and $x_{0,j}$, $x_{f,k}$ are given real constants. Furthermore, linear constraints are imposed on the binary control vector

$$\mathbf{l}_{\min} \le \mathbf{A}\omega \le \mathbf{l}_{\max}, \;\; \mathbf{A} \in \mathbb{R}^{n_A \times n_\omega}, \;\; \mathbf{l}_{\min},\mathbf{l}_{\max} \in \mathbb{R}^{n_A}. \quad (8)$$

The solutions to the MIOCP are the optimal (open loop) trajectories of $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, $0 \le t \le t_f$, the optimal phase transition times t_i^{c*}, the possibly free final time t_f^* and the optimal binary parameter vector ω^*.

Remark 1. A general class of hybrid optimal control problems is defined by determing the optimal hybrid — i.e., continuous $\mathbf{u}$ and discrete $\mathbf{v} : [0,t_f] \to \mathcal{V} \subset$

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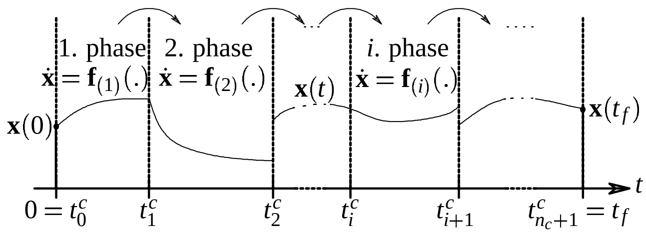


Figure 1: Continuous state dynamics defined in multiple phases, phase transitions occur at switching times t_i^c.

$\mathbb{Z}^{n_v}$ — control trajectories such that a hybrid cost index $\int_0^{t_f} L(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v})\,dt$ is minimized subject to the system dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t)$ and further constraints, where $\mathbf{x}$ denotes the continuous state and $\mathbf{q} : [0,t_f] \to Q \subset \mathbb{Z}^{n_q}$ the discrete state [6]. Assuming a finite, either given or bounded, number of switchings for both $\mathbf{q}$ and $\mathbf{v}$, then the hybrid optimal control problem can be transformed into a MIOCP with integer variables which may be represented by binary variables [6].

Remark 2. The "nature" of the binary control vector ω appearing in MIOCP is ambivalent. On the one hand it represents the discrete control variable $\mathbf{v}$ that controls the order and types of phase transitions, on the other hand it also represents the discrete state $\mathbf{q}$ in each phase.

2 Decomposition Using Branch and Bound and Sparse Direct Collocation

No general solution techniques to address the class of MIOCPs are currently available.

A framework for modeling and (optimally) controlling mixed logical dynamical systems described by linear dynamic equations subject to linear inequalities involing real and integer variables has been proposed by [4]. The on-line optimization problems resulting from a predictive control scheme are solved numerically by application of a mixed-integer quadratic programming branch-and-bound method. However, the approach is not applicable to our class of MIOCPs with *nonlinear* dynamics equations subject to *nonlinear* constraints.

By discretizing the continuous state and control variables of MIOCPs, the resulting mixed-integer nonlinearly constrained optimization problems (MINLPs) are in generally nonconvex. Consequently, the recently developed numerical methods for convex MINLPs [1, 9] cannot be applied, since the bounding properties of the relaxed problem cannot be achieved [2]. However, there is another reason why MINLP techniques are not suited at all: The actual value of the discrete variable determines the sequence, type and number of phase dynamics. Thus, the actual dynamics in a phase and even the dimension or number of constraints may be completely different for different values of the discrete variable. A discretization of the continuous variable, however, must depend on the dynamics and constraints to be any effective or defined at all. This fact has found almost no observation in the literature yet!

A prerequisite for the numerical solution of MIOCPs is the ability to efficiently solve optimal control problems with nonlinear dynamical equations defined in multiple phases subject to nonlinear constraints. The recently developed sparse direct collocation method DIRCOL (Sect. 2.2) satisfies these requirements.

A naive solution approach to MIOCP is to enumerate the feasible discrete control space, i.e., determining all $\omega \subset \{0,1\}^{n_\omega}$ satisfying Eq. (8), and solving all the resulting "continuous" multi-phase optimal control problems related to each of the binary control values. However, even for moderate dimensions of ω, this approach is not feasible because of the NP-completeness of the discrete optimization problem (cf. example of Sect. 3.1).

2.1 Decomposition Using Branch and Bound

The motivation for a decomposition approach is to potentially avoid an explicit enumeration of the entire feasible discrete control space $\{0,1\}^{n_\omega}$ by solving sequences of problems providing rigorous upper (nonincreasing) and lower (nondecreasing) bounds on the MIOCP performance index that converge in a significantly smaller number of iterations.

To obtain an upper bound, the components of ω are set to a combination of 0 or 1 satisfying the linear constraints (8). Hereby, the MIOCP is reduced to a "continuous", multi-phase optimal control problem (primal problem), whose solution is assumed and its global optimum yields a rigorous upper bound on the MIOCP performance index. A lower bound is obtained by the global optimal solution to a multi-phase optimal control problem for a binary control vector with partially relaxed components $0 \le \omega_i \le 1$, $i \in \{1,2,\ldots,n_\omega\}$.

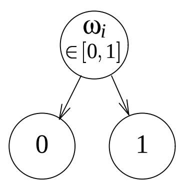


Figure 2: Branching at an inner node of the binary search tree.

A branch and bound technique may start from a root of the binary search tree, where all binary variables are relaxed if the underlying MIOCP permits it. In each inner node some of the binary variables are 0 or 1, all others are relaxed (Fig. 2). At the ends of the search tree, all binary variables are 0 or 1. At an inner node, the global solution of the corresponding multiphase optimal control problem yields a lower bound on the achievable performance index for all nodes of the subtree. If the lower bound in a node is greater than the current best upper bound of the whole search tree, then all subsequent branches from this node can be cut off. Thus, the efficiency of the B&B binary tree search strongly depends on good lower and upper bounds, especially on a good initial upper bound, i.e., a good initial estimate of ω^*.

Various strategies for B&B exist on how to select the branching variable (such as first free variable or maximum fractional part) and how to explore the tree (such as depth first or breadth first). Currently, it is not clear yet which strategy performs best for which MIOCP.

Remark 3. B&B for the binary control vector requires existence of solutions to relaxed MIOCPs, or more precisely, the existence of continuous relaxations to MIOCP. For some MIOCPs, relaxations may exist naturally, although the relaxed problem doesn't need to be of any physical significance with respect to the underlying application. Usually additional modeling effort will be required in defining suitable "meta"-MIOCPs allowing useful relaxations analogously to the definition of superstructures for mixed-integer nonlinear programming problems [1].

Remark 4. As it is to be expected that some modeling effort for the MIOCP must be made before applying numerical methods, it has been suggested to derive suitably simplified and problem specific "screening models" [2]. A screening model can be solved to simultaneously guarantee global optimality and to yield a rigorous lower bound on the solution to the MIOCP, thus avoiding the need for dealing with relaxed MIOCPs. An application for a simple batch process development has successfully been investigated in [2].

Remark 5. The B&B binary tree search incorporates a high parallelism that can be employed in future investigations on a massively parallel computing platform as the communication between processors can be neglected when compared to the computational time for a multi-phase optimal control problem in a node.

Remark 6. The linear constraints (8) on the binary control vector are used during the B&B approach to eliminate nodes before solving any relaxed MIOCPs.

Remark 7. The challenge in solving relaxed MIOCPs during the binary tree search cannot be underestimated. There is no numerical method available that solves optimal control problems with nonlinear dynamics defined in multiple phases and subject to nonlinear constraints and with phase transitions at unknown times *guaranteed* to the global optimum or that even guarantees a locally optimal solution at all. However, not only the global optimum is of interest. For many types of MIOCPs even a "good" solution obtained by the proposed approach that significantly improves the initial guess will be highly appreciated.

2.2 Sparse Direct Collocation for Multi-Phase Optimal Control Problems

The "continuous" multi-phase optimal control problems that are obtained from Eqs. (1) – (8) by having some components of ω fixed to 0 or 1 and some relaxed still include several discrete event effects. For example, the order of phase transitions is given but not the times of switchings. Further switching points also describe the times when a state or control variable inequality constraint (3) becomes active or inactive during a phase.

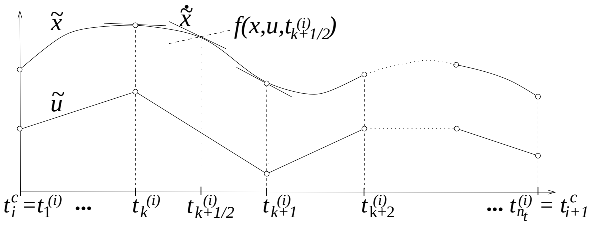


Figure 3: Direct collocation parameterization of continuous state and control variables.

Numerical optimal control methods based on the Euler-Lagrange differential equations (EL-DEQs) and the Maximum Principle (MP) can mainly be divided into two classes: direct and indirect methods [16]. Indirect methods approximate a solution by explicitly solving first and second order optimality conditions resulting from EL-DEQs and the MP. For reasons already discussed in [6, 12, 16] they are not flexible enough for the purpose needed here.

Direct methods are based on a transcription of optimal control problems into (finite dimensional) nonlinearly constrained optimization problems (NLPs) either by direct shooting or direct collocation [5, 16]. Direct methods promise high flexibility and robustness when solving optimal control problems numerically to low or moderate accuracies.

However, in many practical applications the problem functions have only low, local differentiability properties, i.e., discontinuities in the first or second derivatives. Thus, obtaining a useful gradient approximation for shooting-type discretizations is much more involved, since a numerical sensitivity analysis of initial value problems with switching points must be carried out, e.g., [2]. On the other hand, for a collocation-type discretization, only a careful, but much cheaper finite difference approximation is sufficient with usually no need for special treatment of discontinuities in first or second derivatives by switching functions. Additionally appealing in the direct collocation approach is the potentially faster computation compared to direct shooting because the ODE simulation (1) and the control optimization problems (2), (3) are solved simultaneously for collocation and not iteratively as for shooting. To achieve full speed-up for collocation, the NLP sparsity must be fully utitilized. Otherwise the NLP size will severly limit the efficiency.

We apply a discretization of $\mathbf{x}$ by piecewise cubic Hermite polynomials $\tilde{\mathbf{x}}(t) = \sum_j \alpha_j \hat{\mathbf{x}}_j(t)$ and of $\mathbf{u}$ by piecewise linear functions $\tilde{\mathbf{u}}(t) = \sum_k \beta_k \hat{\mathbf{u}}_k(t)$ on a discretization grid $t^c_i = t^{(i)}_1 < t^{(i)}_2 < \ldots < t^{(i)}_{n^{(i)}_t} = t^c_{i+1}$ in each phase (Fig. 3) [12]. The equations of motions (1) are pointwise fulfilled at the grid points and at their respective midpoints resulting in a set of nonlinear NLP equality constraints $\mathbf{a}(\mathbf{y}) = 0$ (collocation at Lobatto points). Any control or state variable inequality constraints are to be satisfied at the grid points resulting in set of nonlinear NLP inequality constraints $\mathbf{b}(\mathbf{y}) \geq 0$. Here, $\mathbf{y}$ denotes the n_y parameters of the parameterization

$$\mathbf{y} = (\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ..., p, t_1^c, \ldots, t_{n_c}^c, t_f)^T.$$

where $p_i \in [0,1], i = 1, \ldots, n_p$ denotes the subset of relaxed binary variables. The resulting nonlinearly constrained optimization problem basically reads as

$$\text{NLP:} \quad \min_{\mathbf{y}} \phi(\mathbf{y}) \quad \text{subject to} \quad \mathbf{a}(\mathbf{y}) = 0,\ \mathbf{b}(\mathbf{y}) \geq 0,$$

where ϕ denotes the parameterized cost index (2).

A carefully selected discretization $\tilde{\mathbf{u}}$, $\tilde{\mathbf{x}}$ must satisfy certain convergence properties. One requirement is that the discretized solution must approximate a solution of the EL-DEQs and the Maximum Principle if the grid becomes fine enough, i.e., for $n_t^{(i)} \to \infty$ and $\max\{t_{k+1}^{(i)} - t_k^{(i)} : k = 1, \ldots, n_t^{(i)} - 1\} \to 0$, cf. [12]. Another great advantage of the direct collocation approach is that it provides reliable estimates $\tilde{\lambda}$ of the adjoint variable trajectory along the discretization grid. These estimates are derived from the Lagrange multipliers of the NLP [12]. They enable a verification of optimality conditions of the discretized solution although the EL-DEQs have not been solved explicitly.

Local optimality error estimates can be derived that enable efficient strategies for successively refining a first solution on a coarse grid [12, 14]. Thus, a sequence of related NLPs must be solved whose dimensions increase with the number of grid points.

NLPs can be solved most efficiently numerically by SQP methods. In each SQP iteration a current guess of the solution $\mathbf{y}^*$ is improved by the solution of a quadratic subproblem derived from a quadratic approximation of the Lagrangian of the NLP subject to the linearized constraints [3, 8].

The NLPs resulting from a direct collocation discretization have several special properties [15]: The NLPs are of large-scale with very many variables and very many constraints. Most of the NLP constraints are active at the solution, e.g., the equality constraints from collocation. Thus, the number of "free" NLP variables is much smaller than the total number of variables n_y. The NLP Jacobians $(\nabla \mathbf{a}(\mathbf{y}), \nabla \mathbf{b}(\mathbf{y}))$ are sparse and structured. Only a few percent of the elements will be nonzero, and the percentage decreases as the number of grid points increases. The NLP objective $\phi(\mathbf{y})$ only depends on a few, fixed number of variables, independently of the actual grid size, if the objective (2) is of Mayer type, i.e., $L \equiv 0$.

All these features can be utilized by the recently developed DIRCOL [14] which applies the recent large-scale SQP method SNOPT [8] which partitions the NLP variables into basic, superbasic and nonbasic variables. The Hessian of the NLP Lagrangian is approximated by limited-memory quasi-Newton updates and a reduced Hessian algorithm is used for solving the QP subproblems. The null-space matrix of the working set in each iteration is obtained from a sparse LU factorization.

The computational speedup achievable by fully utilizing the NLP structure is more than a factor of one hundred for typical discretized optimal control problems when compared to standard "dense" SQP methods [15].

Remark 8. DIRCOL [14] is especially suited for solving the relaxed MIOCPs because of its exceptional robustness and efficiency. Typically only a crude initial guess of parts or all of the solution trajectories of a relaxed MIOCP can be provided. This holds especially for the phase transition times which may have quite different positions at the solution than as they can be provided initially. The large movements of events from their initial to their final position during the course of the optimization method usually pose high difficulties for other methods. On the other hand, an initially crude solution estimate on a rather coarse discretization grid is not a handicap to DIRCOL but the usual way how the solution procedure begins. Finally, a relaxed MIOCP doesn't need to be solved to the finest grid needed for a given tolerance. If the optimality error estimate of the performance index provided by DIRCOL is taken into account, then the computational effort for solving for a refined grid can be avoided if the error estimate indicates that the result for the current grid cannot be below the current upper bound.

3 Hybrid Optimal Control Problems for Traveling Salesmen

In this section we introduce and discuss two new benchmark problems for hybrid optimal control. Their major benefits are that they intuitively help to understand the basic problems encountered in hybrid, mixed-integer optimal control, and that they are easily scalable in terms of their combinatorial complexity while the continuous complexity remains moderate.

3.1 The motorized traveling salesman problem

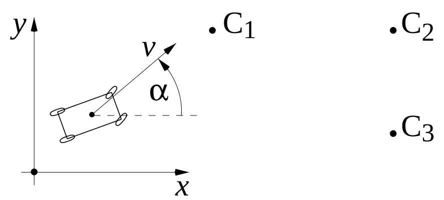
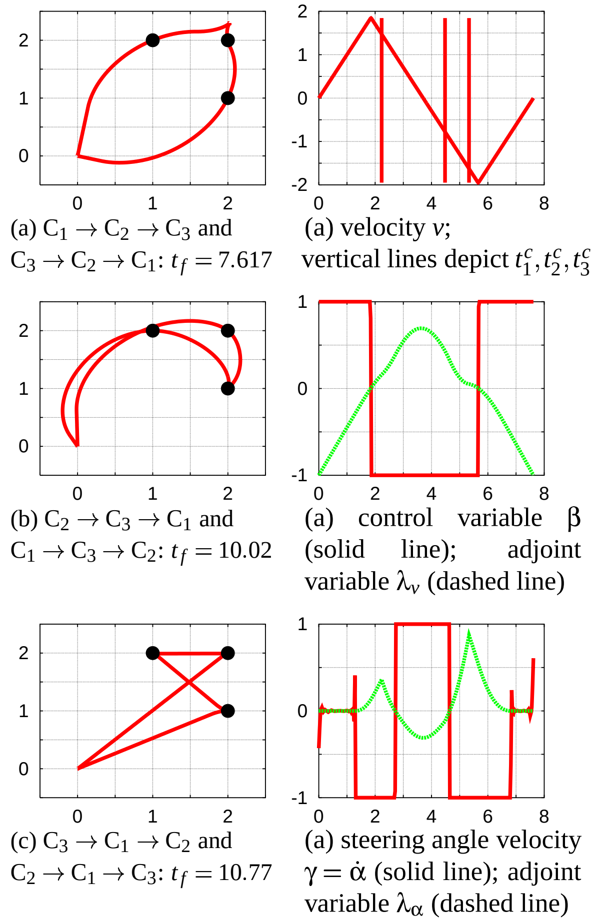


Figure 4: The motorized traveling salesman.

A salesman spends his time visiting n_c cities cyclically. In one tour he passes by each city just once and finishes up in the origin where he started. In what order should he visit them to minimize the overall travel time? The Traveling Salesman Problem (TSP) is one of the most prominent members of combinatorial optimization problems [7, 11]. Here, we introduce a hybrid dynamical extension of the TSP to demonstrate the strong interaction of continuous and discrete dynamics in hybrid optimal control [15] being first presented to the scientific public in [13]. The salesman is supposed to drive a car (Fig. 4). The task is to determine the steering angle velocity γ and the accelerating or braking force β (continuous controls) and the order (discrete control) in which the n_c cities $C_k = (x_k^c, y_k^c)^T$ have to be visited such that the overall travel time is minimized. There are no further restrictions on the path, i.e., the "road", in the (x, y)-plane.



(a) $C_1 \to C_2 \to C_3$ and $C_3 \to C_2 \to C_1$: $t_f = 7.617$

(a) velocity v; vertical lines depict t_1^c, t_2^c, t_3^c

(b) $C_2 \to C_3 \to C_1$ and $C_1 \to C_3 \to C_2$: $t_f = 10.02$

(a) control variable β (solid line); adjoint variable λ_v (dashed line)

(c) $C_3 \to C_1 \to C_2$ and $C_2 \to C_1 \to C_3$: $t_f = 10.77$

(a) steering angle velocity $\gamma = \dot\alpha$ (solid line); adjoint variable λ_α (dashed line)

Figure 5: Left: minimum time tour candidates for 3 cities in the (x, y)-plane; minimum time problem solution is (a). Right: selected state, control and adjoint variables for (a).

A simplified kinematical model of the car is given by

$$\begin{array}{rclll} \dot{x}(t) & = & v(t)\cos(\alpha(t)), & x(0) = 0 = x(t_f), & \\ \dot{y}(t) & = & v(t)\sin(\alpha(t)), & y(0) = 0 = y(t_f), & \\ \dot{v}(t) & = & \beta(t), & v(0) = 0 = v(t_f), & (9) \\ \dot{\alpha}(t) & = & \gamma(t), & \alpha(0),\ \alpha(t_f)\ \text{free}, & \\ |\beta(t)| & \le & 1, & |\gamma(t)| \le 1. & \end{array}$$

A phase describes the travel between two cities. Thus, the number of phases is $n_c + 1$. Let t_i^c denote the time when the i-th city is passed. Then the motorized TSP (MTSP) is formulated as a MIOCP according to Sect. 1 with $\mathbf{u} = (\gamma, \beta)^T$, $\mathbf{x} = (x, y, v, \alpha)^T$, $\omega \in \{0,1\}^{n_c \times n_c}$, and

$$\min_{\mathbf{u},\,\omega} J[\mathbf{u}, \omega] := t_f \qquad (10)$$

$$r_{(i)}(\mathbf{x}(t_i^c - 0), \mathbf{x}(t_i^c + 0), \omega, t_i^c)$$

$$:= \begin{pmatrix} x(t_i^c - 0) \\ y(t_i^c - 0) \end{pmatrix} - \sum_{k=1}^{n_c} \omega_{i,k} \begin{pmatrix} x_k^c \\ y_k^c \end{pmatrix} \qquad (11)$$

$$\mathbf{x}(t_i^c + 0) = R_{(i)}(\mathbf{x}(t_i^c - 0), \omega, t_i^c - 0) := \mathbf{x}(t_i^c - 0) \qquad (12)$$

$$\sum_{i=1}^{n_c} \omega_{i,k} = 1, \quad \sum_{k=1}^{n_c} \omega_{i,k} = 1, \quad 0 \le \omega_{i,k} \le 1. \qquad (13)$$

The latter constraints ensure that each city is visited exactly once on each tour. Each tour, i.e., each feasible ω is a permutation of n_c cities. The problem is autonomous and without active state constraints. Thus, MTSP is symmetric as a tour driven forward or backwards yields the same travel time. Therefore, the number of possible tours is $(n_c)!/2$. The number of tours increases not polynomially with the number of cities. For example, for 3 cities the number of tours is 3, for 5 cities it is 60, for 10 cities it is 1 814 400, and for 50 cities it is approximately $\approx 1.52 \times 10^{64}$. Now if we assume that all tours for 5 cities can be computed in one second inclusive the selection of the best one, then to solve the problem for 20 cities in this way by total enumeration will need approximately $20!/5! \approx 2.03 \times 10^{16}\,\text{s} \approx 643$ million years. The TSP is NP-complete!

For $n_c = 3$ cities $C_1 = (1, 2)$, $C_2 = (2, 2)$, $C_3 = (2, 1)$, the three possible tours are displayed in Fig. 5 (left). For each of the tours the continuous controls and switching times have been optimized using DIRCOL (Sect. 2.2) with respect to the terminal time for a given discrete variable, i.e., order of cities, i.e., sequence of phases. There are $2^{n_c \cdot n_c} = 512$ possible combinations for ω and the binary search tree consists of $2^{n_c \cdot n_c + 1} - 1 = 1023$ elements, but only $(n_c)! = 6$ being feasible with respect to (13). Because of the symmetry only three remain.

The estimates of the adjoint variables provided by DIRCOL are used to check the computed solution (Fig. 5, left (a)) for consistency to the EL-DEQs and the Maximum Principle: Both controls appear linearly in the Hamiltonian $H := \lambda_x v \cos\alpha + \lambda_y v \sin\alpha + \lambda_v \beta + \lambda_\alpha \gamma$. Thus, $\lambda_v > 0 \Rightarrow \beta = -1$ and $\lambda_v < 0 \Rightarrow \beta = +1$ are conditions satisfied by the computed solution (Fig. 5, middle right). An analogous argument holds for γ and λ_α (Fig. 5, bottom right). In addition, two intervals $[0, 1.28]$, $[6.82, t_f]$ of singular control γ appear where $\lambda_\alpha(t) = 0$. The existence of these two singular control intervals is not surprising. Because $\alpha(0)$ and $\alpha(t_f)$ are free, $\lambda_\alpha(0) = \partial\varphi/\partial\alpha(0) = 0 = \partial\varphi/\partial\alpha(t_f) = \lambda_\alpha(t_f)$ must hold.

3.2 Two cooperating motorized traveling salesman

Two salesmen start their tours in the origin and return to it after each city has been visited just once by one and only one of them. Which of the cities and in what order should each of the salesmen visit them to minimize the overall travel time? We consider a hybrid, cooperative dynamic game extension of the MTSP for two salesmen representing the class of rather autonomous dynamic agents that cooperate optimally [10, 13, 15].

The solution for 5 cities is displayed in Fig. 6. There

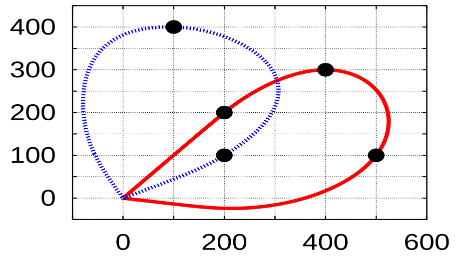


Figure 6: The tours of two cooperating motorized salesmen to 5 cities.

exist $2\sum_{i=1}^{n_c} i! = 306$ possible tours. A problem formulation with 36 binary variables [10, 13, 15] results in a binary search tree with $2^{37} - 1 = 1.374... \times 10^{11}$ nodes, but only 306 combinations being feasible with respect to the corresponding linear constraints (8). The computation for B&B with a depth-first strategy takes 2 hours and 45 minutes to solve for 151 nodes on a Linux-PC with a Pentium III/500 MHz processor. The problem solution time for a single, relaxed MIOCP varied between a few seconds and three minutes although the average solution time per problem was about one minute.

The minimum time objective only determines the solution for one salesman uniquely, and t_f^* only gives an upper bound for the travel time of the second salesman. If the second one is additionally required to minimize the energy subject to an overall tour time less or equal t_f^*, then the trajectories of the second one will also be (locally) uniquely determined.

4 Conclusions

A first method to solving quite general hybrid optimal controls is presented. A decomposition approach is applied to efficiently solving fairly general MIOCPs by branch-and-bound and sparse direct collocation. The decomposition relies on the capabilites of the recently developed, sparse direct collocation method DIRCOL for efficiently solving optimal control problems with multivariable nonlinear dynamics defined in multiple phases subject to nonlinear constraints. Furthermore, two new benchmark hybrid optimal control problems are presented for a single and two cooperating motorized traveling salesmen.

Focus of further research will be the derivation of better lower bounds, e.g., from a dualization of a hybrid maximum principle, and, thus, a more efficient binary search. This may also help in extending the sparse direct collocation method to attain global solutions in case of multiple local minima. Further methods are needed to determine good initial estimates of the discrete variables yielding good upper bounds and possibly relaxing the need for a global binary search. Finally, towards synthesizing feedback controllers for MIOCPs the development of theories for perturbation and sensitivity analysis of such problems are needed. Despite the encouraging first results, research in numerical hybrid optimal control is still at the beginning.

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