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FACHGEBIET SIMULATION UND SYSTEMOPTIMIERUNG (SIM)

User's Guide for

DIRCOL

A Direct Collocation Method
for the Numerical Solution of
Optimal Control Problems

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Abstract

This report forms the user's guide for Version 2.1 of ***DIRCOL***, a set of Fortran subroutines designed to solve optimal control problems of systems described by (first order) ordinary differential equations subject to general inequality or equality constraints on the control and/or state variables. Discontinuities in the right hand side of the differential equations can be treated as well as multi-phase problems. The user must provide subroutines that define the objective and the constraint functions. Subroutines for derivatives of the problem functions are not required.

DIRCOL is a direct collocation method [39]: By a discretization of state and control variables the infinite dimensional optimal control problem is transcribed into a sequence of (finite dimensional) nonlinearly constrained optimization problems (NLPs). Optimal Control Theory and adjoint differential equations are not required in order to apply the algorithm. The NLPs are solved either by the dense Sequential Quadratic Programming (SQP) method NPSOL (Gill, Murray, Saunders, and Wright [12]) or by the sparse SQP method SNOPT (Gill, Murray, and Saunders [11], Version 5.3-4 or higher).

DIRCOL also computes reliable estimates of the adjoint variables, the multiplier functions of state constraints and the switching structure. Therefore the method can conveniently be combined with an indirect method such as multiple shooting [40].

Supplementary programs are provided for supporting a visualization of the numerical results.

A Note for Potential Users

The user's guide does **not** replace courses in Optimal Control Theory and Numerical Analysis. As any other numerical method for optimal control, ***DIRCOL*** will be used most efficiently by a user having background knowledge in Optimal Control Theory and Numerical Analysis, especially, numerical solution of ordinary differential equations and numerical nonlinearly constrained optimization.

Bugs and Comments

Please report bugs or comments to `stryk@informatik.tu-darmstadt.de`. Also any reports on experiences (good or bad), suggestions for improvements or complaints concerning the program or this user's guide are wellcome!

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New!!

☞ Significant changes as well as new options in both the program and the user's guide newly available with Versions 2.0 and 2.1 when compared to Versions 1.2 and 1.3 are emphasized in the text by a mark *New!!* at the left or right margin of the corresponding page of the user's guide.

1 Purpose

DIRCOL is a collection of Fortran 77 subroutines for solving *optimal control problems*. The code is supplied with a main program. To tackle an optimal control problem numerically using ***DIRCOL***, already prepared problem dependent subroutines and input files can be used (see Section 4.2 for details).

Before the general description of optimal control problems treatable by the algorithm is given in Section 1.2, a basic problem description in Section 1.1 introduces the problem statement.

1.1 Basic optimal control problem

Find the *control variable* $u(t)$, the *control* (or *design*) *parameter* p, and the possibly free final time t_f that minimize the functional (i e., the *objective* or the *performance index*)

$$J[u, t_f] \;=\; \Phi(x(t_f), t_f) \tag{1.1}$$

subject to the *state equation*

$$\dot{x}(t) \;=\; \frac{\mathrm{d}x(t)}{\mathrm{d}t} \;=\; f(x(t), u(t), p, t), \quad t_0 \le t \le t_f, \tag{1.2}$$

where the initial time t_0 is given and fixed. The function f is called the *right hand side* of the differential equations.

The *state variable* $x(t)$ has to satisfy *initial* and *final* (*terminal*) *conditions*, i. e., conditions at initial and final time,

$$x(t_0) = x_0, \quad x(t_f) = x_f\,. \tag{1.3}$$

An inequality constraint on the state variable (*state constraint*)

$$g(x(t)) \;\ge\; 0 \tag{1.4}$$

or a *control constraint*

$$g(x(t), u(t)) \;\ge\; 0 \tag{1.5}$$

may also have to be satisfied for all $t \in [t_0, t_f]$.

1.2 General problem formulation

In the general problem formulation, piecewise defined right hand sides or constraints can be treated by *switching points* t_S that may be free or fixed. Then the whole time interval can be splitted up into several *phases*. The phases are connected by phase connecting conditions imposed on the left and right side limits of the state variables at the switching point and on the switching point itself.

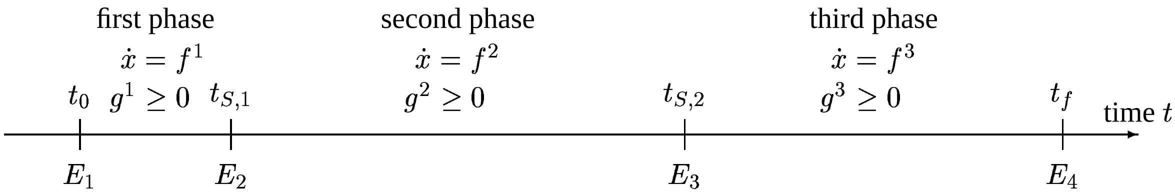


Figure 1.1: A problem with $m_1 = 3$ phases and $m = 4$ events.

For example, consider a problem with three phases as depicted in Figure 1.1 and a piecewise defined right hand side $f^i(x(t), u(t), p, t)$, $i = 1, 2, 3$. The switching points $t_{S,1}$ and $t_{S,2}$ and the initial and final times can be collected into a vector of m *events* E by $E_1 = t_0$, $E_2 = t_{S,1}$, $E_3 = t_{S,2}$, $E_4 = t_f$. In this example, there are $m = 4$ events, $m_1 := m - 1 = 3$ phases, and $m - 2 = 2$ switching points. In general, the vector of events reads as

$$\begin{aligned} E &:= (E_1, E_2, \ldots, E_{m-1}, E_m) \\ &:= (t_0, t_{S,1}, \ldots, t_{S,m-2}, t_f). \end{aligned} \tag{1.6}$$

Hint: Use problem formulations with a single phase $[E_1, E_m] = [t_0, t_f]$, $m = 2$, $m_1 = 1$, whenever possible! Introducing switching points complicates the problem formulation and increases the possibility of errors by the user in the correct problem programming and use of ***DIRCOL*** enormously. Using multi-phase problem formulations is only recommended to really advanced users.

Notation: For the general problem formulation, the following **definitions** and **notations** are used

$$x(t) = \begin{pmatrix} x_1(t) \\ \dots \\ x_{n_X}(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ \dots \\ u_{l_U}(t) \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ \dots \\ p_{l_P} \end{pmatrix}.$$

Constant controls p (so-called *control parameters* or *design parameters*) are treated separately from the time dependent variables u in the general problem formulation.

$[t_0, t_f]$	:	the time interval considered: $t_0 < t_f$, $[t_0, t_f] \subset \mathbb{R}$
t_0	:	the fixed initial time
t_f	:	the fixed or free final time
$x,\ x(t)$	:	the state variables, $x : [t_0, t_f] \to \mathbb{R}^{n_X}$
$u,\ u(t)$	:	the control variables, $u : [t_0, t_f] \to \mathbb{R}^{l_U}$
p	:	the control (or design) parameters, $p \in \mathbb{R}^{l_P}$
E	:	the m events: $E_1 := t_0, \dots, E_m := t_f$
n_X	:	number of state variables ($n_X \geq 1$)
l_U	:	number of control variables ($l_U \geq 0$)
l_P	:	number of control parameters ($l_P \geq 0$)
m	:	number of events ($m \geq 2$)
m_1	:	$m_1 := m-1 \geq 1$ is the total number of phases $[E_i, E_{i+1}],\ i = 1, \dots, m_1$
Φ	:	the real valued objective function (to be minimized)
f	:	the right hand side of the differential equations $\dot{x}(t) = f(x(t), u(t), p, t)$
f^i	:	the right hand side of the differential equations $\dot{x}(t) = f^i(x(t), u(t), p, t)$ in the i-th phase: $E_i \leq t \leq E_{i+1}$, $i \in \{1, \dots, m_1\}$
g	:	all nonlinear inequality constraints $g(x(t), u(t), p, t) \geq 0$
g^i	:	all nonlinear inequality constraints $g^i(x(t), u(t), p, t) \geq 0$ in the i-th phase: $E_i \leq t \leq E_{i+1}$, $i \in \{1, \dots, m_1\}$
g^i_k	:	the k-th nonlinear inequality constraint $g^i_k(.) \geq 0$ in the i-th phase
$n_{g,nln,i}$	:	the number of nonlinear inequality constraints $g^i_k(.) \geq 0$, $k = 1, \dots, n_{g,nln,i}$, in the i-th phase, $i = 1, \dots, m_1$, ($n_{g,nln,i} \geq 0$)
h	:	all nonlinear equality constraints $h(x(t), u(t), p, t) = 0$
h^i	:	all nonlinear equality constraints $h^i(x(t), u(t), p, t) = 0$ in the i-th phase: $E_i \leq t \leq E_{i+1}$, $i \in \{1, \dots, m_1\}$
h^i_k	:	the k-th nonlinear equality constraint $h^i_k(.) = 0$ in the i-th phase
$n_{h,nln,i}$	:	the number of nonlinear equality constraints $h^i_k = 0$, $k = 1, \dots, n_{h,nln,i}$, in the i-th phase, $i = 1, \dots, m_1$ ($n_{h,nln,i} \geq 0$)
r^i	:	the implicit boundary or phase connecting (i. e., switching) conditions
$n_{r,nln,i}$	:	$i = 1$: the number of implicit boundary conditions at t_0, t_f $i = 2, \dots, m_1$: the number of implicit phase connecting conditions at the switching point $t_{S,i-1} = E_i$ ($n_{r,nln,i} \geq 0$)
$x(t-0)$	:=	$\lim_{\epsilon>0,\ \epsilon\to 0} x(t-\epsilon)$ (left side limit of x at t)
$x(t+0)$	:=	$\lim_{\epsilon>0,\ \epsilon\to 0} x(t+\epsilon)$ (right side limit of x at t)

In the sequel, references to user-supplied input files and subroutines are given in combination with the problem formulation. The details of the files that have to be supplied by the user are described later on in Section 4.2.

1.2.1 Objective

In general, the real-valued Mayer-type objective Φ may depend on

- values of the state variables x at the initial time t_0 and/or at the final time t_f and/or on the left and/or right side limits of x at the switching points,
- the final time t_f itself and (or) the switching points (collected in the vector of events E), and
- the control (or design) parameters p, i. e.,

$$\begin{aligned} J[u,p] \; = \; \Phi\Big(& x(t_0), \\ & x(t_{S,1}-0), t_{S,1}, x(t_{S,1}+0), \\ & \ldots, \\ & x(t_{S,m-2}-0), t_{S,m-2}, x(t_{S,m-2}+0), \\ & x(t_f), t_f, \, p\Big). \end{aligned} \tag{1.7}$$

New!!

In the current version of ***DIRCOL***, the additional assumption on the structure of Φ has been made for simplification of the user-supplied subroutine USROBJ for computing Φ (see Section 4.2.2.3)

$$\Phi \; = \; \Phi_1 + \ldots + \Phi_i + \ldots + \Phi_{m_1} \; = \; \sum_{i=1}^{m_1} \Phi_i, \tag{1.8}$$

$$\begin{aligned} \text{where} \quad \Phi_1 & = \Phi_1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) \\ & = \Phi_1(x(E_1), u(E_1), p, E_1, x(E_m), u(E_m), E_m), \\ \text{and} \quad \Phi_{i+1} & = \Phi_i(x(t_{S,i}-0), u(t_{S,i}-0), p, t_{S,i}, x(t_{S,i}+0), u(t_{S,i}+0), t_f) \\ & = \Phi_i(x(E_{i+1}-0), u(E_{i+1}-0), p, E_{i+1}, x(E_{i+1}+0), u(E_{i+1}+0), E_m), \\ & \quad i = 1, \ldots, m-2. \end{aligned}$$

This assumption may be dropped. However, this will require a few changes in the source code of ***DIRCOL***.

1.2.2 Differential equations

The first order ordinary differential equations may be piecewise defined by

$$\dot{x}(t) = \begin{cases} f^1(x(t), u(t), p, t), & t \in [t_0, t_{S,1}] = [E_1, E_2], \\ f^i(x(t), u(t), p, t), & t \in [t_{S,i-1}, t_{S,i}] = [E_i, E_{i+1}], \\ & \qquad i = 2, \ldots, m-2, \\ f^{m-1}(x(t), u(t), p, t), & t \in [t_{S,m-2}, t_f] = [E_{m-1}, E_m], \end{cases} \tag{1.9}$$

where the right hand side in each phase is

$$f^i(x, u, p, t) = \begin{cases} f_1^i(x, u, p, t) \\ \ldots \\ f_{n_X}^i(x, u, p, t) \end{cases}, \text{ i. e., } f^i : \mathbb{R}^{n_X + l_U + l_P + 1} \to \mathbb{R}^{n_X}, \quad i = 1, \ldots, m-1.$$

The right hand side $f_k^i,\ k = 1, \ldots, n_X,\ i = 1, \ldots, m_1$, of each phase has to be provided by the user-supplied subroutine USRDEQ (see Section 4.2.2.2).

1.2.3 Conditions at initial, final and switching times

Explicit and *implicit* formulations of boundary and phase connecting conditions are distinguished.

Explicit conditions will **always** be fulfilled at every iteration of the optimization method because the discretization of the optimal control problem is tailored to them.

Implicit conditions will usually only be fulfilled at the end of an optimization run but **not** at intermediate iterations as they appear as nonlinear equality constraints in the nonlinear optimization problem.

Hint: Whenever there is a choice in the problem formulation, an *explicit* formulation should be preferred over an *implicit* one.

Note: Although it is possible by the software to impose conditions on the control variables $u(t)$ at initial and final time and at switching points, this usually doesn't make sense and may even be **harmful** concerning the existence of a solution of the optimal control problem and its computability. Thus, be careful when using this option.

1.2.3.1 Explicit formulation

Two kinds of explicit formulations are distinguished in this user's guide.

The constant values of the *first kind* (concerning the values of the state variables at initial time $x(t_0)$ and of the left side limits of the state variables at the switching points $x(t_{S,k}-0)$, $k=1,\ldots,m-2$) have to be provided by the user-supplied input file DATLIM (see Section 4.2.1.2).

The functions of the *second kind* (concerning the values of the state variables at the final time $x(t_f)$ and of the right side limits of the state variables at the switching points $x(t_{S,k}+0)$, $k=1,\ldots,m-2$) have to be provided by the user-supplied subroutine USRNBC (see Section 4.2.2.4).

Information about each type of conditions has to be provided by the user-supplied input file DATLIM (see Section 4.2.1.2).

1.2.3.1.1 Explicit formulation of the first kind: The variables at *initial time* as well as their *left side limits* at *switching points* may be set explicitly to a given constant (provided by the user-supplied input file DATLIM, Section 4.2.1.2) as

$$
\begin{array}{lcll}
x_j(t_0) & = & x_{j,0} & \text{(a given constant)} \\
u_l(t_0) & = & u_{l,0} & \text{(a given constant)} \\[1ex]
x_j(t_{S,k}-0) & = & x_{j,S,k} & \text{(a given constant)} \\
u_l(t_{S,k}-0) & = & u_{l,S,k} & \text{(a given constant)} \\
 & & & k=2,\ldots,m_1 \text{ (only possible if } m_1 \geq 2) \\[1ex]
\Big(j \in \{1,\ldots,n_X\}, & & & l \in \{1,\ldots,l_U\}\Big).
\end{array}
\tag{1.10}
$$

1.2.3.1.2 Explicit formulation of the second kind: The variables at *final time* as well as their *right side limits* at *switching points* may be set explicitly to a given constant *or* a function R of the values at initial time or of the left side limits at switching points, respectively,

$$
\begin{aligned}
x_j(t_f) &= R^1_{x,j}(x(t_0), u(t_0), p, t_0, t_f) \\
u_l(t_f) &= R^1_{u,l}(x(t_0), u(t_0), p, t_0, t_f) \\
x_j(t_{S,k}+0) &= R^{k+1}_{x,j}(x(t_{S,k}-0), u(t_{S,k}-0), p, t_{S,k}) \\
u_l(t_{S,k}+0) &= R^{k+1}_{u,l}(x(t_{S,k}-0), u(t_{S,k}-0), p, t_{S,k}) \\
& \quad k = 2, \ldots, m_1 \text{ (only possible if } m_1 \geq 2) \\
\Big(j \in \{1, \ldots, n_X\}, &\qquad l \in \{1, \ldots, l_U\}\Big).
\end{aligned}
\tag{1.11}
$$

Some examples of such functions R^k_i are

$$
\begin{aligned}
x_j(t_{S,k}+0) &= 100.0 && \text{(a given constant)}, \\
x_j(t_{S,k}+0) &= x_j(t_{S,k}-0) && \text{(continuity of } x_i(t) \text{ at } t_{S,k}), \\
x_j(t_f) &= x_j(t_0) && \text{(periodicity of } x_i(t)).
\end{aligned}
\tag{1.12}
$$

The functions $R^k_{x,j}$ and $R^k_{u,l}$ have to be provided by the user-supplied subroutine USRNBC (see Section 4.2.2.4). Information about each type of conditions has to be provided by the user-supplied input file DATLIM (see Section 4.2.1.2).

1.2.3.2 Implicit formulation

$$
\begin{aligned}
r^1_i(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) &= 0 \\
i = 1, \ldots, n_{r,nln,1} \quad (n_{r,nln,1} \geq 0) & \\
r^k_l(x(t_{S,k-1}-0), u(t_{S,k-1}-0), p, t_{S,k-1}, x(t_{S,k-1}+0), u(t_{S,k-1}+0), t_f) &= 0 \\
l = 1, \ldots, n_{r,nln,k} \quad (n_{r,nln,k} \geq 0) & \\
\text{for} \quad k = 2, \ldots, m_1. &
\end{aligned}
\tag{1.13}
$$

New!!

The functions r defining implicit boundary and phase connecting conditions have to be provided by the user-supplied subroutine USRNBC (see Section 4.2.2.4). The numbers of each type of conditions have to be provided by the user-supplied input file DATDIM (see Section 4.2.1.1).

1.2.4 Inequality constraints

In order to improve robustness and efficiency, *upper* and *lower bounds* on all variables are treated separately from *general inequality constraints*.

1.2.4.1 Upper and lower bounds

Upper and lower bounds (so-called *box constraints*) have to be provided for **all** of the state variables x, the control variables u, the control parameters p, and the events E.

The box constraints of state and control variables may differ from one phase to another

$$\begin{array}{llrclcl} \text{1st phase:} & t \in [E_1, E_2], & x^1_{j,\min} & \leq & x_j(t) & \leq & x^1_{j,\max}, \quad j = 1, \ldots, n_X, \\ & & u^1_{l,\min} & \leq & u_l(t) & \leq & u^1_{l,\max}, \quad l = 1, \ldots, l_U, \\ \ldots & & & & & & \\ m_1\text{-th phase:} & t \in [E_{m_1}, E_m], & x^{m_1}_{j,\min} & \leq & x_j(t) & \leq & x^{m_1}_{j,\max}, \quad j = 1, \ldots, n_X, \\ & & u^{m_1}_{l,\min} & \leq & u_l(t) & \leq & u^{m_1}_{l,\max}, \quad l = 1, \ldots, l_U. \end{array} \tag{1.14}$$

The box constraints of the control parameters read as

$$p_{l,\min} \;\leq\; p_l \;\leq\; p_{l,\max}, \quad l = 1, \ldots, l_P, \tag{1.15}$$

and for the events

$$\begin{array}{rcccl} E_{2,\min} & \leq & E_2 & \leq & E_{2,\max}, \\ & & \ldots & & \\ E_{m,\min} & \leq & E_m & \leq & E_{m,\max}. \end{array} \tag{1.16}$$

The initial time $E_1 = t_0$ is kept fixed. Therefore no box constraints on t_0 are needed.

- All upper and lower bounds have to be provided by the user-supplied input file DATLIM (see Section 4.2.1.2).
- **Unconstrained** variables: If a variable has no upper bound, i. e., is unconstrained from above, then a value greater or equal to +1.E+10 $= +10^{10} \cong +\infty$ (which is treated as a placeholder for infinity) has to be set as upper bound. Vice versa, if a variable has no lower bound, a value less or equal to $-$1.E+10 $= -10^{10} \cong -\infty$ has to be specified in DATLIM (cf. optional parameter *infinite bound size* of NPSOL [12] or optional parameter *infinite bound* of SNOPT [11]).
- Problems with a **fixed final time** or **fixed switching points** are treated by setting lower and upper bounds of the corresponding event to the same value. For example, for a fixed final time of $t_f = 9.0$ one sets

$$E_{m,\min} = 9.0 \quad \text{and} \quad E_{m,\max} = 9.0\,. \tag{1.17}$$

1.2.4.2 Nonlinear inequality constraints

Number and type of the nonlinear inequality constraints may differ from one phase to another

$$\begin{array}{lllll} \text{1st phase:} & t \in [E_1, E_2], & g_k^1(x,u,p,t) \geq 0, & k = 1,\ldots,n_{g,nln,1} & (n_{g,nln,1} \geq 0) \\ \ldots & & & & \\ m_1\text{-th phase:} & t \in [E_{m_1}, E_m], & g_k^{m_1}(x,u,p,t) \geq 0, & k = 1,\ldots,n_{g,nln,m_1} & (n_{g,nln,m_1} \geq 0) \end{array} \tag{1.18}$$

The real-valued constraint functions g_k^i have to be provided by the user-supplied subroutine USRNIC (see Section 4.2.2.5). The numbers $n_{g,nln,i}$, $i = 1,\ldots,m_1$ have to be provided by the user-supplied input file DATDIM (see Section 4.2.1.1).

1.2.5 Equality constraints

Number and type of the nonlinear equality constraints may differ from one phase to another

$$\begin{array}{lllll} \text{1st phase:} & t \in [E_1, E_2], & h_k^1(x,u,p,t) = 0, & k = 1,\ldots,n_{h,nln,1} & (n_{h,nln,1} \geq 0) \\ \ldots & & & & \\ m_1\text{-th phase:} & t \in [E_{m_1}, E_m], & h_k^{m_1}(x,u,p,t) = 0, & k = 1,\ldots,n_{h,nln,m_1} & (n_{h,nln,m_1} \geq 0) \end{array} \tag{1.19}$$

The real-valued constraint functions h_k^i have to be provided by the user-supplied subroutine USRNEC (see Section 4.2.2.6). The numbers $n_{h,nln,i}$, $i = 1,\ldots,m_1$ have to be provided by the user-supplied input file DATDIM (see Section 4.2.1.1).

1.3 Other problem formulations

Several other problem formulations can be transformed into the problem formulation of Section 1.2.

1.3.1 Bolza and Lagrange-type objectives

Bolza and Lagrange-type objectives with an integral term can be transformed to Mayer-type problems by introducing an additional state variable.

For example, consider the Bolza problem with the objective

$$J[u, t_f] \;=\; \Phi(x(t_f), t_f) \;+\; \int_{t_0}^{t_f} L(x(t), u(t), t)\mathrm{d}t. \tag{1.20}$$

The Mayer form is obtained by introducing x_{n_X+1} and one additional state equation

$$\dot{x}_{n_X+1}(t) \;=\; L(x(t), u(t), t), \quad x_{n_X+1}(t_0) \;=\; 0, \quad x_{n_X+1}(t_f) \text{ free}, \tag{1.21}$$

in order to obtain the "new" objective

$$J[u, t_f] \;=\; \Phi(x(t_f), t_f) \;+\; x_{n_X+1}(t_f) \;=\; \Phi^\star(x^\star(t_f), t_f) \tag{1.22}$$

with the "new" state variable $x^\star = (x_1, \ldots, x_{n_X}, x_{n_X+1})^T$ which is of dimension $n_X{}^\star = n_X + 1$.

Examples: This transformation is used in the minimum energy problem of Section 2.2 and the oscillator problem of Section 2.6.

1.3.2 Integral-type constraints

Integral-type constraints

$$\int_{t_0}^{t_f} G(x(t), u(t), t)\mathrm{d}t \;=\; 0 \quad (\text{or} \;\geq\; 0) \tag{1.23}$$

can be treated in the same way as integral-type objectives by introducing an additional state variable

$$\dot{x}_{n_X+1}(t) \;=\; G(x(t), u(t), t), \quad x_{n_X+1}(t_0) \;=\; 0 \tag{1.24}$$

resulting in the "new" end point constraint

$$x_{n_X+1}(t_f) \;=\; 0 \quad (\text{or} \;\geq\; 0). \tag{1.25}$$

1.3.3 Min-max objectives

Min-max objectives such as

$$J[u,p] \;=\; \min_{u,p} \max_{t\in[t_0,t_f]} M(x(t),u(t),t), \tag{1.26}$$

can be transformed into constrained Mayer-type problems by introducing an additional control parameter p_{l_P+1} satisfying $p_{l_P+1} = \max\{M(x(t),u(t),t) : t \in [t_0,t_f]\}$. Then the resulting constrained Mayer-type problem is

$$J[u,p^\star] = \Phi^\star(x(t_f),p^\star,t_f) = p_{l_P+1} \;\rightarrow\; \text{min!} \tag{1.27}$$

subject to the additional inequality constraint

$$p_{l_P+1} - M(x(t),u(t),t) \;\geq\; 0. \tag{1.28}$$

Example: This transformation is used in the problem of the flight of a passenger aircraft through a downburst windshear described in Section 2.8.

1.3.4 Higher order differential equations

Higher order differential equations may be transformed to first order systems by introducing additional state variables.

For example, consider the second order system

$$\ddot{x} \;=\; f(x,u,p,t). \tag{1.29}$$

This system is equivalent to the first order system

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= f(x,u,p,t) \end{aligned} \tag{1.30}$$

which is of the double size. The "new" state variable is $x^\star = (x,v)$.

Example: This transformation technique is applied to the minimum energy problem of Section 2.2 and to the robot trajectory optimization problems of Section 2.9.

1.3.5 DAEs and implicit differential equations

Implicit differential equations and differential-algebraic equations (DAEs) may be transformed to explicit first order systems with equality constraints by introducing additional control variables.

For example, consider the **first order implicit system**

$$h(x, \dot{x}, u, p, t) \;=\; 0. \tag{1.31}$$

This implicit system is equivalent to the first order system with nonlinear equality constraints

$$\begin{aligned} \dot{x} &= v, \\ 0 &= h(x, v, u, p, t). \end{aligned} \tag{1.32}$$

The "new" control variable is $u^\star = (u, v)$.

Another example are **semi-explicit DAEs** of the form

$$\begin{aligned} \dot{x} &= f(x, y, u, p, t), \\ 0 &= h(x, y, u, p, t) \end{aligned} \tag{1.33}$$

where dimension(f) = dimension$(x) = n_X$ and dimension(h) = dimension$(y) = n_Y$, i. e., x denote the differential state and y the algebraic state variables. For the numerical solution using ***DIRCOL*** the algebraic equation $h(.) = 0$ ist treated as nonlinear equality constraint (see Section 1.2.5) and the algebraic state variable y is considered to be a control variable. Thus, the "new" control variable is $u^\star = (u, y)$ of dimension $l_{U^\star} = l_U + n_Y$. Thus, the algebraic state variable y is approximated by piecewise linear functions.

Examples are the index-1 DAE formulation of the catalyst mixing problem described in Section 2.4.3 and the index-3 DAE formulation of the pendulum problem described in Section 2.5.

2 Example Problems

In this chapter the formulations of the example problems are described, which are provided in the subdirectories of `dircol-2.0/ex` with the source code.

2.1 Survey of the example problems

problem/directory	n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{g,nln,2}$	$n_{h,nln,1}$	$n_{h,nln,2}$	$n_{r,nln,1}$	$n_{r,nln,2}$
`minimum_energy`	3	1	0	1						
(standard)					0	—	0	—	0	—
`_demo`					1	—	0	—	2	—
`onedim_rocket`	2	1	0	2						
(standard)					0	0	0	0	0	0
`_demo`					1	1	0	0	3	2
`catalyst_mixing`										
`_ode`	2	1	0	1	0	—	0	—	0	—
`_dae`	2	2	0	1	0	—	1	—	0	—
`pendulum`	4	1	1	1	0	—	1	—	2	—
`oscillator`	5	0	2	1	0	—	0	—	1	—
`robot_i2`	5	2	0	1	0	—	0	—	0	—
`windshear`	5	1	1	1	1	—	0	—	0	—
`manutec`										
`_minimum_time`	6	3	0	1	0	—	0	—	0	—
`_minimum_energy`	7	3	0	1	0	—	0	—	0	—

Table 2.1: Dimensions of the example problems.

Brief comments on the example problems:

1. *Minimum energy:*
   A classical minimum energy problem from Bryson, Denham, and Dreyfus with a second order state constraint.

2. *One-dimensional rocket:*
   Ascent optimization of a simple rocket in two phases.

3. *Catalyst mixing*:
   Optimal mixing policy of two catalysts of a plug flow reactor (index-1 DAE).

4. *Pendulum*:
   Parameter identification for a pendulum (index-3 DAE, multiple solutions).

5. *Oscillator:*
   Optimal design parameters for minimum noise design of a microwave oscillator.

6. *Roboter i2:*
   Optimal path tracking for a simple robot model.

7. *Windshear:*
   Min-max optimal control problem of abort landing of a passenger aircraft in the presence of a downburst windshear subject to several constraints.

8. *Manutec minimum time:*
   The minimum time point-to-point trajectory of an industrial robot with three degrees-of-freedom whose dynamics is described realistically by a highly nonlinear system of differential equations. The trajectory is subject to (active) box constraints on the state variables, i. e., the angles and the angular velocities.

9. *Manutec minimum energy:*
   A minimum energy trajectory for the same robot with prescribed final time.

For all example problems a reference for the solution is available either as an explicit solution, another approximate solution, or an ideal reference path.

Remark: Many of the example problems are discussed in detail in [39].

2.2 A minimum energy problem

2.2.1 Problem description

This well-known problem has originally been suggested by John V. Breakwell and its analytical solution has been reported by Bryson, Denham, and Dreyfus [5]. It is also discussed in Sec. 3.11, Example 2, of Bryson, Ho [6] and has been investigated in Sec. 5.3.1 of [39]. The task is to minimize

$$J[u] = \frac{1}{2}\int_0^1 a^2(t)\mathrm{d}t \tag{2.1}$$

subject to the differential equation with boundary conditions

$$\begin{array}{rclrclrcl} \ddot{x}(t) & = & a(t), & x(0) & = & 0, & x(1) & = & 0, \\ & & & \dot{x}(0) & = & 1, & \dot{x}(1) & = & -1 \end{array} \tag{2.2}$$

and the constraint

$$x(t) \le l \tag{2.3}$$

for a given constant l.

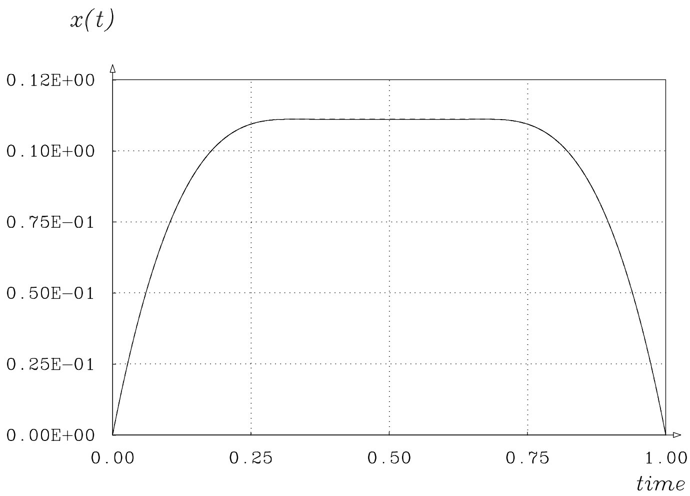


Figure 2.1: Optimal state $x(t)$ of the minimum energy problem in the case of an upper bound of $l = 1/9 = 0.1111\ldots$

2.2.2 `minimum_energy`: the standard formulation

According to the terminology of Section 1.2 the Lagrange-type objective has to be transformed into a Mayer-type one by introducing a third state variable. Obviously, the problem consists of one phase only, i. e., $m_1 = 1$ and $[E_1, E_2] = [t_0, t_f] = [0, 1]$. According to Section 1.2 we have the following dimensions

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
3	1	0	1	0	0	0

The state and control variables are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \frac{1}{2}\int_0^t a^2(\tau)\mathrm{d}\tau \end{pmatrix}, \quad \begin{pmatrix} u_1 \end{pmatrix} = \begin{pmatrix} a \end{pmatrix}. \tag{2.4}$$

New!!

The objective according to Section 1.2.1 is

$$\begin{aligned} \Phi &= \Phi_1 \\ \text{where}\quad \Phi_1 &= x_3(t_f). \end{aligned} \tag{2.5}$$

According to Section 1.2.2, the right hand side of the differential equations now reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \end{pmatrix} = \begin{pmatrix} x_2 \\ u_1 \\ \frac{1}{2}u_1^2 \end{pmatrix}. \tag{2.6}$$

The conditions at the initial time $t_0 = 0$ are explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{aligned} x_1(t_0) &= 0, \\ x_2(t_0) &= 1, \\ x_3(t_0) &= 0. \end{aligned} \tag{2.7}$$

The conditions at the final time $t_f = 1$ are explicit conditions of the second kind according to Equation (1.11) of Section 1.2.3.1

$$\begin{aligned} x_1(t_f) &= R_{x,1}^1(.) = 0, \\ x_2(t_f) &= R_{x,2}^1(.) = -1, \\ x_3(t_f) &\ \text{is free.} \end{aligned} \tag{2.8}$$

As the upper and lower bounds of Section 1.2.4.1 one obtains for the state and control variables in the

$$\begin{array}{lllllllll} \text{1st phase:} \quad t \in [E_1, E_2], & -10^{10} & = & x_{1,\min}^1 & \le & x_1(t) & \le & x_{1,\max}^1 & = l = \frac{1}{9}, \\ & -10^{10} & = & x_{2,\min}^1 & \le & x_2(t) & \le & x_{2,\max}^1 & = +10^{10}, \\ & -10^{10} & = & x_{3,\min}^1 & \le & x_3(t) & \le & x_{3,\max}^1 & = +10^{10}, \\ & -10^{10} & = & u_{1,\min}^1 & \le & u_1(t) & \le & u_{1,\max}^1 & = +10^{10} \end{array} \tag{2.9}$$

Control parameters do not appear in this problem (i. e., $l_P = 0$) and the final time is fixed. Therefore the lower and upper bounds for E_2 are

$$1 = E_{2,\min} \le E_2 \le E_{2,\max} = 1. \tag{2.10}$$

Nonlinear inequality or equality constraints as described in Sections 1.2.4 and 1.2.5 do not appear in this problem: $n_{g,nln,1} = 0$, $n_{h,nln,1} = 0$.

2.2.3 `minimum_energy_demo`: formulation for demonstration

For the purpose of demonstration only, another, usually less efficient formulation of the problem for numerical solution is also provided:

- Here, the upper bound constraint on x_1 is treated as a nonlinear inequality constraint according to Section 1.2.4.2. Therefore $n_{g,nln,1} = 1$ and

$$g_1^1(x, u, p, t) \;=\; l - x_1(t) \;\geq\; 0. \tag{2.11}$$

- The two explicit conditions at the final time $t_f = 1$ of Equation (2.8) are replaced by two implicit conditions according to Equation (1.13) of Section 1.2.3.2

$$\begin{array}{rclcl} r_1^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) & = & x_1(t_f) & = & 0, \\ r_2^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) & = & x_2(t_f) + 1 & = & 0. \end{array} \tag{2.12}$$

For the new formulation of the problem, the dimensions according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
3	1	0	1	1	0	2

In order to investigate the effects of the new problem formulation, the explicit upper bound on $x_1(t)$ is set to infinity:

$$\text{1st phase:} \quad t \in [E_1, E_2], \quad -10^{10} \;=\; x_{1,\min}^1 \;\leq\; x_1(t) \;\leq\; x_{1,\max}^1 \;=\; +10^{10}.$$

Everything else in the problem formulation remains the same compared with the standard formulation of the previous section.

2.3 One-dimensional ascent of a rocket

2.3.1 Problem description

This simple one-dimensional rocket problem with an analytical solution has been given by Hargraves [17]. The rocket has no drag and constant thrust a until a time t_S of phase separation and it has zero thrust after this time.

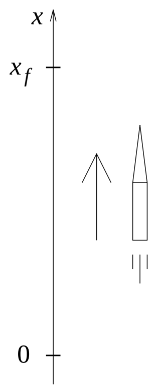


Figure 2.2: The one-dimensional ascent of a simple rocket.

The problem is to reach a given altitude x_f with minimum fuel consumption. Assuming a constant fuel flow rate, fuel consumption is directly proportional to t_S. Thus we want to minimize the time t_S of phase separation while the final time t_f is free.

The original problem formulation [17] reads as: Minimize

$$J[\eta, t_S, t_f] = t_S \tag{2.13}$$

subject to the differential equation

$$\ddot{x} = \eta(t)a - g \tag{2.14}$$

and boundary conditions

$$\begin{array}{rclcrcl} x(0) & = & 0, & & x(t_f) & = & x_f \\ \dot{x}(0) & = & v_0, & & \dot{x}(t_f) & \text{is} & \text{free.} \end{array} \tag{2.15}$$

The control is given by

$$\eta(t) = \begin{cases} 1, & t \in [0, t_S] \\ 0, & t \in (t_S, t_f] \end{cases} \tag{2.16}$$

and the constants are $g = 1$, $a = 2$, $x_f = 100$, and $v_0 = 0$.

Several formulations of the problem according to Section 1.2 are possible, for example,

- as a problem with one control variable and different control constraints in different phases, or
- by eliminating the control η a priori using Equation (2.16) and resulting in a problem with a phase-wise defined right hand side (2.15).

Here, we will discuss the first formulation.

2.3.2 `onedim_rocket`: the standard formulation

The problem has $m_1 = 2$ phases, and the $m = 3$ events are

$$E_1 = 0,\ E_2 = t_S,\ E_3 = t_f. \tag{2.17}$$

Hence, the dimensions according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{g,nln,2}$	$n_{h,nln,1}$	$n_{h,nln,2}$	$n_{r,nln,1}$	$n_{r,nln,2}$
2	1	0	2	0	0	0	0	0	0

The state and control variables are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix}, \quad \begin{pmatrix} u_1 \end{pmatrix} = \begin{pmatrix} \eta \end{pmatrix}. \tag{2.18}$$

The objective according to Section 1.2.1 is

New!!

$$\begin{array}{rcl} \Phi & = & \Phi_1 + \Phi_2 \\ \text{where}\ \Phi_1 & = & 0, \quad \Phi_2 \ = \ E_2. \end{array} \tag{2.19}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 \ = \ \begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix} = \begin{pmatrix} x_2 \\ au_1 - g \end{pmatrix}, \quad E_1 \le t \le E_2, \tag{2.20}$$

$$f^2 \ = \ \begin{pmatrix} f_1^2 \\ f_2^2 \end{pmatrix} = \begin{pmatrix} x_2 \\ au_1 - g \end{pmatrix}, \quad E_2 < t \le E_3. \tag{2.21}$$

The conditions at the initial time $E_1 = 0$ are explicit conditions of the first kind in the terminology of Section 1.2.3

$$\begin{array}{rcl} x_1(t_0) & = & 0, \\ x_2(t_0) & = & 0. \end{array} \tag{2.22}$$

The condition at the final time $E_3 = t_f$ is an explicit condition of the second kind in the terminology of Section 1.2.3

$$x_1(t_f) \ = \ R_{x,1}^1(.) \ = \ x_f. \tag{2.23}$$

Although it is not explicitly mentioned in the problem description, the state variables x_1 and x_2 are *continuous* at the staging point $E_2 = t_S$. We have to ensure this for the discretized state variables by explicit conditions of the second kind according to Equation (1.11) of Section 1.2.3.1

$$\begin{array}{rcccl} x_1(E_2+0) & = & R_{x,1}^2(.) & = & x_1(E_2-0), \\ x_2(E_2+0) & = & R_{x,2}^2(.) & = & x_2(E_2-0). \end{array} \tag{2.24}$$

As the upper and lower bounds of Section 1.2.4.1 for the state and control variables we obtain

$$\begin{array}{llrclclclcl} \text{1st phase:} & t \in [E_1, E_2], & 0 & = & x_{1,\min}^1 & \le & x_1(t) & \le & x_{1,\max}^1 & = & +10^{10}, \\ & & 0 & = & x_{2,\min}^1 & \le & x_2(t) & \le & x_{2,\max}^1 & = & +10^{10}, \\ & & 1 & = & u_{1,\min}^1 & \le & u_1(t) & \le & u_{1,\max}^1 & = & 1, \\ \text{2nd phase:} & t \in [E_2, E_3], & 0 & = & x_{1,\min}^2 & \le & x_1(t) & \le & x_{1,\max}^2 & = & +10^{10}, \\ & & 0 & = & x_{2,\min}^2 & \le & x_2(t) & \le & x_{2,\max}^2 & = & +10^{10}, \\ & & 0 & = & u_{1,\min}^2 & \le & u_1(t) & \le & u_{1,\max}^2 & = & 0. \end{array} \tag{2.25}$$

As the lower and upper bounds for E_2 and E_3

$$\begin{array}{ccccccccc} 0.01 & = & E_{2,\min} & \leq & E_2 & \leq & E_{2,\max} & = & +10^{10}, \\ 0.01 & = & E_{3,\min} & \leq & E_3 & \leq & E_{3,\max} & = & +10^{10}, \end{array} \tag{2.26}$$

might be used.

2.3.3 `onedim_rocket_demo`: formulation for demonstration

For the purpose of demonstration only, another, usually less efficient formulation of the problem for numerical solution is also provided:

- The two explicit conditions at initial time of Equation (2.22), the explicit condition at final time of Equation (2.23), and the two explicit conditions at the point of phase separation of Equation (2.24) are replaced by implicit conditions:

$$\begin{array}{rclcl} r_1^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) & = & x_1(t_0) & = & 0, \\ r_2^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) & = & x_2(t_0) & = & 0, \\ r_3^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) & = & x_1(t_f) - x_f & = & 0, \\ r_1^2(x(t_S - 0), u(t_S - 0), p, t_S, x(t_S + 0), u(t_S + 0)) & = & x_1(t_S + 0) - x_1(t_S - 0) & = & 0, \\ r_2^2(x(t_S - 0), u(t_S - 0), p, t_S, x(t_S + 0), u(t_S + 0)) & = & x_2(t_S + 0) - x_2(t_S - 0) & = & 0. \end{array} \tag{2.27}$$

- In each phase, one (trivial) nonlinear inequality constraint is introduced which is not active at the solution:

$$\begin{array}{lllllll} \text{1st phase:} & t \in [E_1, E_2], & g_1^1(x, u, p, t) & = & x_1(t) & \geq & 0, \\ \text{2nd phase:} & t \in [E_2, E_3], & g_1^2(x, u, p, t) & = & x_2(t) & \geq & 0, \end{array} \tag{2.28}$$

Hence, the dimensions according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{g,nln,2}$	$n_{h,nln,1}$	$n_{h,nln,2}$	$n_{r,nln,1}$	$n_{r,nln,2}$
2	1	0	2	1	1	0	0	3	2

Everything else in the problem formulation remains unchanged.

2.4 Catalyst mixing

2.4.1 Problem description

New!!

This problem due to Gunn and Thomas [16] (see also [3, 25, 42]) determines the optimal mixing policy of two catalysts along the length of a tubular, plug flow reactor involving the reactions

$$S_1 \rightleftharpoons S_2 \rightarrow S_3 \, .$$

The mixing ratio of the catalysts is the control variable u which has to be determined in order to maximize the production of species S_3, i. e., to minimize

$$J[u] \;=\; -1 + x_1(t_f) + x_2(t_f) \tag{2.29}$$

subject to the differential equations with boundary conditions

$$\begin{array}{rclrcll} \dot{x}_1(t) &=& u(t)(10x_2(t) - x_1(t)), & x_1(0) &=& 1, & x_1(t_f) \text{ free}, \\ \dot{x}_2(t) &=& u(t)(x_1(t) - 10x_2(t)) - (1 - u(t))x_2(t), & x_2(0) &=& 0, & x_2(t_f) \text{ free}, \end{array} \tag{2.30}$$

and the control constraint

$$0 \;\leq\; u(t) \leq 1 \tag{2.31}$$

for fixed final time $t_f = 1$.

A reference value of -0.0480557 for the objective obtained numerically has been reported by [42].

2.4.2 `catalyst_mixing_ode`: the ODE formulation

Obviously, the problem consists of only one phase, i. e., $m_1 = 1$ and $[E_1, E_2] = [t_0, t_f] = [0, 1]$. According to Section 1.2 we have the following dimensions

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
2	1	0	1	0	0	0

The objective according to Section 1.2.1 is

$$\begin{aligned} \Phi &= \Phi_1 \\ \text{where} \quad \Phi_1 &= -1 + x_1(t_f) + x_2(t_f). \end{aligned} \tag{2.32}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix} = \begin{pmatrix} u_1(10x_2 - x_1) \\ u_1(x_1 - 10x_2) - (1 - u_1)x_2 \end{pmatrix}. \tag{2.33}$$

The conditions at the initial time $t_0 = 0$ are explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{aligned} x_1(t_0) &= 1, \\ x_2(t_0) &= 0. \end{aligned} \tag{2.34}$$

No conditions on the state variables are given at the final time $E_2 = t_f$.

As the upper and lower bounds of Section 1.2.4.1 one obtains for the state and control variables in the

$$\begin{aligned} \text{1st phase:} \quad t \in [E_1, E_2], \quad -10^{10} &= x^1_{1,\min} \le x_1(t) \le x^1_{1,\max} = +10^{10}, \\ -10^{10} &= x^1_{2,\min} \le x_2(t) \le x^1_{2,\max} = +10^{10}, \\ 0 &= u^1_{1,\min} \le u_1(t) \le u^1_{1,\max} = +1. \end{aligned} \tag{2.35}$$

Control parameters do not appear in this problem (i. e., $l_P = 0$) and the final time is fixed. Thus, the lower and upper bounds for E_2 are equal

$$1 = E_{2,\min} \le E_2 \le E_{2,\max} = 1. \tag{2.36}$$

Nonlinear inequality or equality constraints as described in Sections 1.2.4 and 1.2.5 do not appear in this problem: $n_{g,nln,1} = 0$, $n_{h,nln,1} = 0$.

2.4.3 `catalyst_mixing_dae`: the DAE formulation

An index-1 DAE formulation of the problem is described by [3] where a third, *algebraic* state variable x_3 is introduced that must satisfy the algebraic equation

$$x_3(t) = 1 - x_1(t) - x_2(t). \tag{2.37}$$

Thus, the objective to be minimized is rewritten as

$$J[u] = -x_3(t_f). \tag{2.38}$$

For the numerical treatment of the reformulated problem using ***DIRCOL***, the algebraic equation is treated as a nonlinear equality constraint according to Section 1.3.5. Thus, the *algebraic state variable* is considered to be another *control variable* u_2 and is approximated piecewise linearly.

The problem still consists of only one phase, i. e., $m_1 = 1$ and $[E_1, E_2] = [t_0, t_f] = [0, 1]$. According to Section 1.2 we have the new problem dimensions

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
2	2	0	1	0	1	0

The objective according to Section 1.2.1 is

$$\begin{aligned} \Phi &= \Phi_1 \\ \text{where} \quad \Phi_1 &= -u_2(t_f). \end{aligned} \tag{2.39}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \end{pmatrix} = \begin{pmatrix} u_1(10x_2 - x_1) \\ u_1(x_1 - 10x_2) - (1 - u_1)x_2 \end{pmatrix}. \tag{2.40}$$

The conditions at the initial time $t_0 = 0$ are explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{aligned} x_1(t_0) &= 1, \\ x_2(t_0) &= 0. \end{aligned} \tag{2.41}$$

No conditions on the state variables are given at the final time $E_2 = t_f$.

As the upper and lower bounds of Section 1.2.4.1 one obtains for the state and control variables in the

$$\begin{array}{lrcccccccl} \text{1st phase:} \quad t \in [E_1, E_2], & -10^{10} & = & x^1_{1,\min} & \le & x_1(t) & \le & x^1_{1,\max} & = & +10^{10}, \\ & -10^{10} & = & x^1_{2,\min} & \le & x_2(t) & \le & x^1_{2,\max} & = & +10^{10}, \\ & 0 & = & u^1_{1,\min} & \le & u_1(t) & \le & u^1_{1,\max} & = & +1, \\ & -10^{10} & = & u^1_{2,\min} & \le & u_2(t) & \le & u^1_{2,\max} & = & +10^{10}. \end{array} \tag{2.42}$$

Control parameters do not appear in this problem (i. e., $l_P = 0$) and the final time is fixed. Thus, the lower and upper bounds for E_2 are equal

$$1 = E_{2,\min} \le E_2 \le E_{2,\max} = 1. \tag{2.43}$$

Nonlinear inequality constraints as described in Section 1.2.4 do not appear in this problem: $n_{g,nln,1} = 0$. However, there is $n_{h,nln,1} = 1$ equality constraint (cf. Section 1.2.5)

$$h_1^1(x, u, p, t) = 1 - x_1(t) - x_2(t) - u_2(t) = 0. \tag{2.44}$$

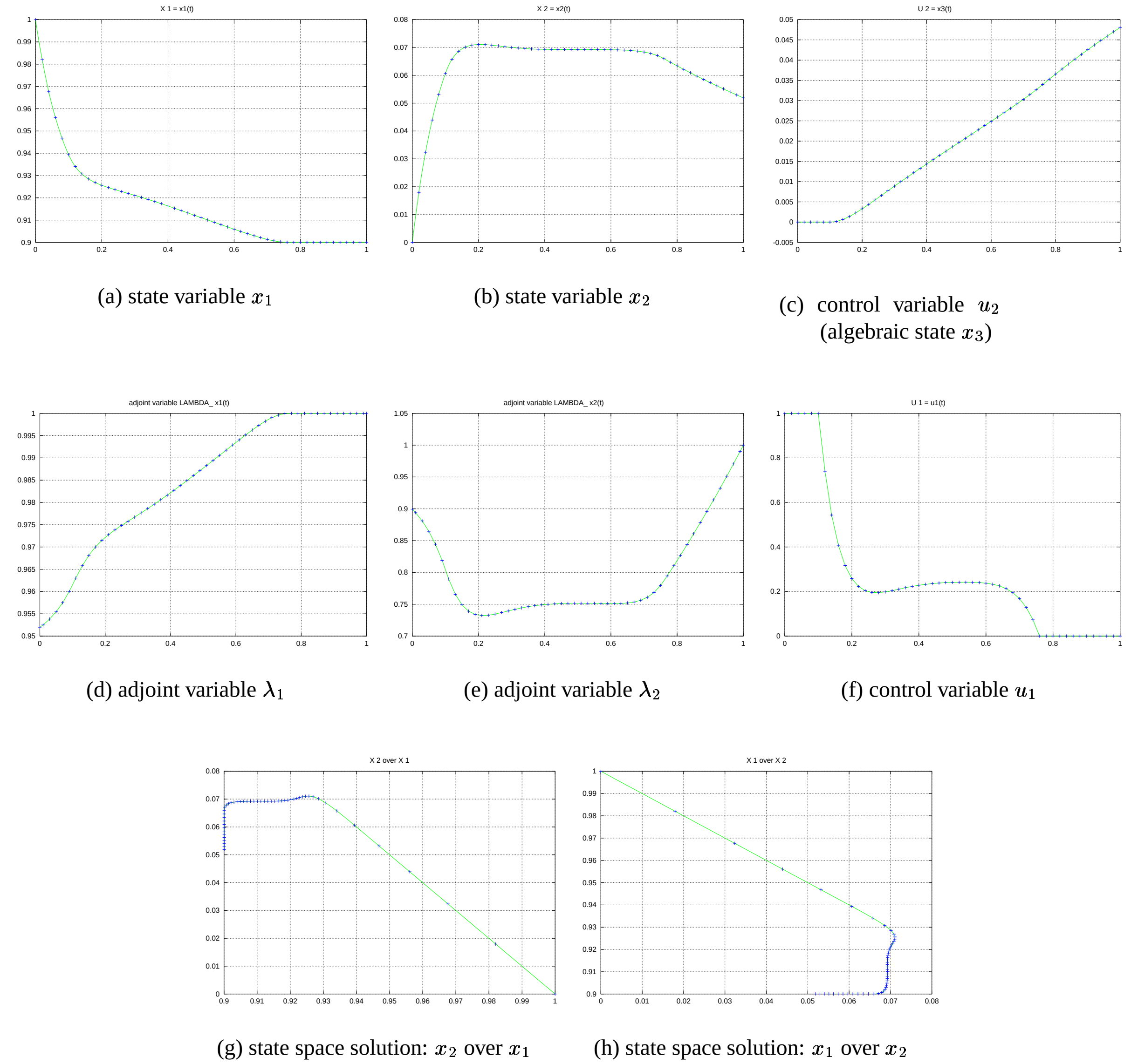
Fig. 2.3 displays the numerical solution of the problem applying ***DIRCOL*** with 51 equidistant grid points and with option `iAction = 8` for optimization (Section 4.2.1.1).

For the initial estimates (cf. Sections 4.2.2.7 and 4.4.1.3)

$$x_1^{\text{initial}}(t) \equiv 1, \quad x_2^{\text{initial}}(t) \equiv 0, \quad u_1^{\text{initial}}(t) \equiv 0, \quad u_2^{\text{initial}}(t) \equiv 0, \quad t \in [0,1]$$

and the tolerances $\epsilon_{\text{OPT}} = 10^{-6}$, $\epsilon_{\text{NFT}} = 10^{-8}$, a good approximation of the solution with a final value of the objective of -0.0480455993 is obtained in very few seconds (for example, in 0.8 seconds with a Pentium II/266 MHz under Linux 2.0.33 with GNU Fortran Version 0.5.19.1).

The numerical results have been visualized using DGNU and Gnuplot (cf. Section 4.6.1). The computed optimal control u_1^* exhibits a max-singular-min structure.



(a) state variable x_1 (b) state variable x_2 (c) control variable u_2 (algebraic state x_3)

(d) adjoint variable λ_1 (e) adjoint variable λ_2 (f) control variable u_1

(g) state space solution: x_2 over x_1 (h) state space solution: x_1 over x_2

Figure 2.3: ***DIRCOL*** solution of the catalyst mixing problem for 51 equistant grid points.

2.5 Pendulum

New!!

2.5.1 Problem description

Consider a pendulum of length $l = 0.5\,[\mathrm{m}]$ and mass $m = 0.3\,[\mathrm{kg}]$ moving in a vertical plane being fixed at the origin of the x-y coordinate system [19, 20].

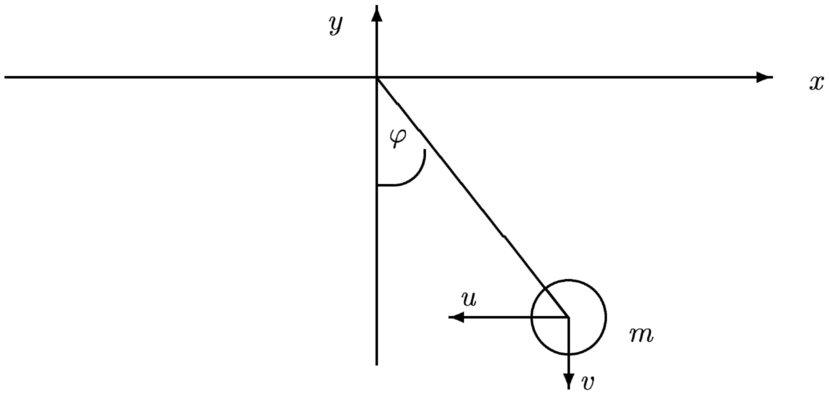


Figure 2.4: State variables of the pendulum.

The equations of motion may be described by a system of DAEs of (differential) index 3 [19, 20]

$$\begin{array}{rclrcrl} \dot{x}(t) &=& u(t) & x(0) &=& 0.4 & [\mathrm{m}] \\ \dot{y}(t) &=& v(t) & y(0) &=& -0.3 & [\mathrm{m}] \\ \dot{u}(t) &=& \lambda x(t)/m & u(0) &=& 0 & [\mathrm{m/s}] \\ \dot{v}(t) &=& \lambda y(t)/m - g & v(0) &=& 0 & [\mathrm{m/s}] \\ 0 &=& x^2(t) + y^2(t) - l^2 & & & & \end{array} \tag{2.45}$$

where λ denotes the Lagrangian multiplier being the algebraic state variable.

We assume that the gravitational constant g is unknown and is to be estimated using the results of an experiment (cf. [19, 20]). As suggested by Kronseder [24], only the result

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} -0.231625 \\ -0.443109 \end{pmatrix} \tag{2.46}$$

of one measurement at $t_f = 2\,[\mathrm{s}]$ is used to obtain an objective of Mayer type

$$J \;=\; \frac{1}{2}\left((x(t_f) - x_f)^2 + (y(t_f) - y_f)^2\right). \tag{2.47}$$

The measurement values are the numerical solution of an initial value problem for the "true" gravitational constant $g^* = 9.81\,[\mathrm{m/s}^2]$ plus a white measurement noise simulating a measurement error.

Index reduction [34] is applied to transform the index 3 into an index 1 system analyticaly by successive total differentiation with respect to time resulting in a "new" algebraic constraint (on the acceleration level)

$$0 \;=\; u^2(t) + v^2(t) + \lambda(t) l^2/m - y(t) g \tag{2.48}$$

plus two additional entry conditions at $t_0 = 0$ (cf. [19, 20, 24])

$$\begin{array}{rcl} 0 &=& x^2(t_0) + y^2(t_0) - l^2 \\ 0 &=& x(t_0)u(t_0) + y(t_0)v(t_0). \end{array} \tag{2.49}$$

☞ **Please note** that the solution to this problem is **not** unique! The pendulum swings periodically. Different numbers of swings already completed at $t_f = 2$ correspond to different solutions for g. Thus, the numerical solution depends on the initial estimates, namely λ^{initial}, g^{initial}.

2.5.2 `pendulum`: the standard formulation

The problem consists of one phase, i. e., $m_1 = 1$ and $[E_1, E_2] = [t_0, t_f] = [0, 2]$.

The state variables, control variable and control parameter are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}, \qquad \begin{array}{l} u = \begin{pmatrix} u_1 \end{pmatrix} = \begin{pmatrix} \lambda \end{pmatrix}, \\ p = \begin{pmatrix} p_1 \end{pmatrix} = \begin{pmatrix} g \end{pmatrix}. \end{array} \tag{2.50}$$

According to Section 1.2 we have the following dimensions

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
4	1	1	1	0	1	2

The objective according to Section 1.2.1 is

$$\begin{array}{rcl} \Phi & = & \Phi_1 \\ \text{where} \quad \Phi_1 & = & \frac{1}{2}\left((x_1(t_f) - x_f)^2 + (x_2(t_f) - y_f)^2\right) \end{array} \tag{2.51}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ u_1(t)x_1(t)/m \\ u_1(t)x_2(t)/m - p_1 \end{pmatrix}. \tag{2.52}$$

The conditions at the initial time $t_0 = 0$ and final time $t_f = 2$ consist of explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{array}{rcr} x_1(t_0) & = & 0.4, \\ x_2(t_0) & = & -0.3, \\ x_3(t_0) & = & 0, \\ x_4(t_0) & = & 0, \end{array} \tag{2.53}$$

no explicit conditions of the second kind according to Equation (1.11) of Section 1.2.3.1, but two implicit conditions according to Equation (1.13) of Section 1.2.3.2

$$r_1^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) = x_1^2(t_0) + x_2^2(t_0) - l^2 = 0, \tag{2.54}$$

$$r_2^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) = x_1(t_0)x_3(t_0) + x_2(t_0)x_4(t_0) = 0. \tag{2.55}$$

As the upper and lower bounds of Section 1.2.4.1 one obtains for the state and control variables in the

$$\begin{array}{llcccccccc} \text{1st phase:} & t \in [E_1, E_2], & -10^{10} & = & x_{1,\min}^1 & \leq & x_1(t) & \leq & x_{1,\max}^1 & = & +10^{10}, \\ & & -10^{10} & = & x_{2,\min}^1 & \leq & x_2(t) & \leq & x_{2,\max}^1 & = & +10^{10}, \\ & & -10^{10} & = & x_{3,\min}^1 & \leq & x_3(t) & \leq & x_{3,\max}^1 & = & +10^{10}, \\ & & -10^{10} & = & x_{4,\min}^1 & \leq & x_4(t) & \leq & x_{4,\max}^1 & = & +10^{10}, \\ & & -10^{10} & = & u_{1,\min}^1 & \leq & u_1(t) & \leq & u_{1,\max}^1 & = & +10^{10}, \end{array} \tag{2.56}$$

and for the control parameter and event

$$\begin{array}{rcccccccc} 10^{-2} & = & p_{1,\min} & \leq & p_1 & \leq & p_{1,\max} & = & 100, \\ 2 & = & E_{2,\min} & \leq & E_2 & \leq & E_{2,\max} & = & 2. \end{array} \tag{2.57}$$

Nonlinear inequality constraints as described in Section 1.2.4 do not appear in this problem: $n_{g,nln,1} = 0$. However, there is $n_{h,nln,1} = 1$ equality constraint (cf. Section 1.2.5)

$$h_1^1(x, u, p, t) \;=\; x_3^2(t) + x_4^2(t) + u_1(t) l^2/m - x_2(t) p_1 \;=\; 0. \tag{2.58}$$

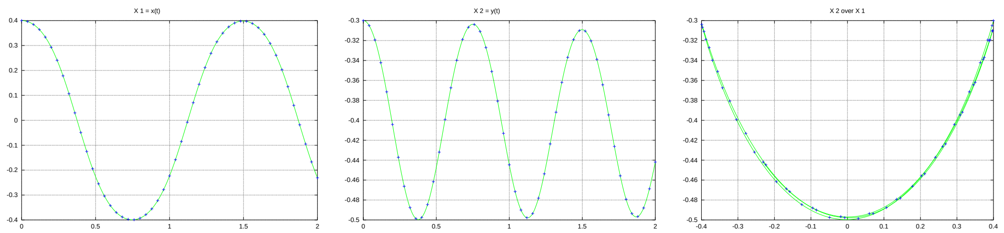
Fig. 2.5 displays the numerical solution of the problem applying ***DIRCOL*** with 51 equidistant grid points and with option `iAction = 8` for optimization (Section 4.2.1.1).

For the initial estimates (cf. Sections 4.2.2.7 and 4.4.1.3) for $t \in [0, 2]$

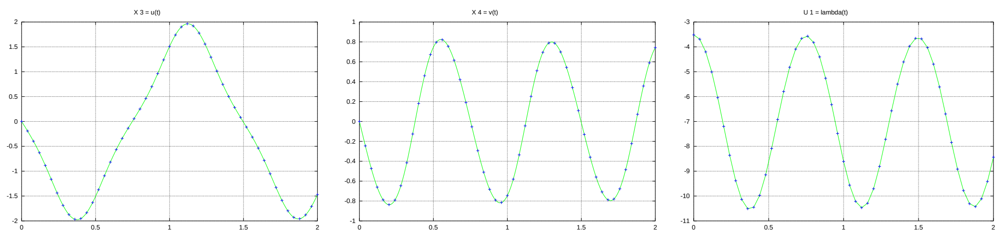
$$x_1^{\text{initial}}(t) \equiv 0.4,\; x_2^{\text{initial}}(t) \equiv -0.3,\; x_3^{\text{initial}}(t) = x_4^{\text{initial}}(t) \equiv 0,\; u_1^{\text{initial}}(t) \equiv -5.0,$$

and $p_1^{\text{initial}} = 20.0$, and the tolerances $\epsilon_{\text{OPT}} = 10^{-6}$, $\epsilon_{\text{NFT}} = 10^{-8}$, a good approximation of the solution with a final value of the objective of $0.955756149 \cdot 10{-}6$ and of the gravitational constant $p_1 = g = 9.7704$ is obtained in very few seconds (for example, in 1.2 CPU seconds with a Pentium II/266 MHz under Linux 2.0.33 with GNU Fortran Version 0.5.19.1).

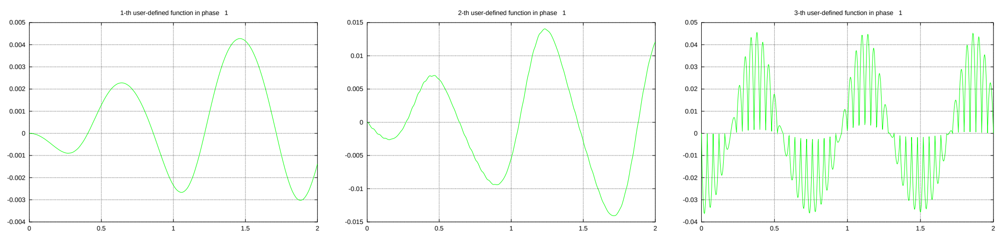
The numerical results have been visualized using DGNU and Gnuplot (cf. Section 4.6.1).



(a) state variable x (b) state variable y (c) state space solution (x, y)



(d) state variable $u = \dot{x}$ (e) state variable $v = \dot{y}$ (f) algebraic variable λ



(g) $x^2(t) + y^2(t) - l^2$ (h) $x(t)u(t) + y(t)v(t)$ (i) $h_1^1 = u^2(t) + v^2(t) + \lambda(t) l^2/m - y(t) g$

Figure 2.5: ***DIRCOL*** solution to the pendulum problem for 51 equistant grid points.

2.6 Minimum noise design of an oscillator

This problem has been suggested to the author by W. Anzill who also reported the analytical solution. It is described in Section 8.6 of [39].

2.6.1 Problem description

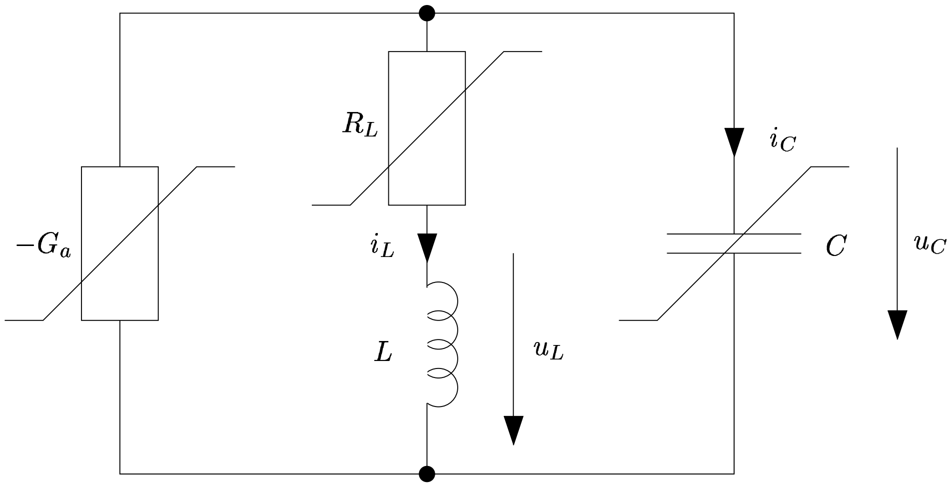


Figure 2.6: Equivalent circuit of the van-der-Pol-oscillator.

The task is to determine the design, e. g., the linear network, of oscillators with minimized phase noise. This is done by formulating and solving an appropriate optimal control problem numerically. Here, we will only briefly describe the model problem of minimum noise design of a van-der-Pol-oscillator which has been described in [1, 39]. For more background on the general problem we refer to [2].

The state variables are

$$x_1 = u_C, \qquad x_2 = \sqrt{L/C} \cdot i_L \tag{2.59}$$

where u_C describes the voltage, L the inductivity, C the capacity. The differential equations are

$$\dot{x}_1 = \frac{\gamma}{2}\left(1 - \frac{x_1^2(t) + x_2^2(t)}{R_0^2}\right) x_1(t) - x_2(t)\left(\omega + \beta\left(\sqrt{x_1^2(t) + x_2^2(t)} - R_0\right)\right) \tag{2.60}$$

$$\dot{x}_2 = \frac{\gamma}{2}\left(1 - \frac{x_1^2(t) + x_2^2(t)}{R_0^2}\right) x_2(t) + x_1(t)\left(\omega + \beta\left(\sqrt{x_1^2(t) + x_2^2(t)} - R_0\right)\right) \tag{2.61}$$

with real constants γ, β, R_0, ω. For computing the single-sideband phase noise two variables v_1 and v_2 are required which must satisfy

$$\begin{aligned}\dot{v}_1 = & -\left(\frac{\gamma}{2}\left(1 - \frac{x_1^2(t) + x_2^2(t)}{R_0^2} - 2\frac{x_1^2(t)}{R_0^2}\right) - \beta\frac{x_1(t)x_2(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}\right) v_1(t) \\ & - \left(-\gamma\frac{x_1(t)x_2(t)}{R_0^2} + \omega + \beta\left(\sqrt{x_1^2(t) + x_2^2(t)} - R_0\right) + \beta\frac{x_1^2(t)}{\sqrt{x_1^2(t) + x_2^2(t)}}\right) v_2(t),\end{aligned} \tag{2.62}$$

$$\dot{v}_2 = -\left(-\gamma\frac{x_1(t)x_2(t)}{R_0^2} - \omega - \beta\left(\sqrt{x_1^2(t)+x_2^2(t)} - R_0\right) - \beta\frac{x_2^2(t)}{\sqrt{x_1^2(t)+x_2^2(t)}}\right) v_1(t)$$
$$-\left(\frac{\gamma}{2}\left(1 - \frac{x_1^2(t)+x_2^2(t)}{R_0^2} - 2\frac{x_2^2(t)}{R_0^2}\right) + \beta\frac{x_1(t)x_2(t)}{\sqrt{x_1^2(t)+x_2^2(t)}}\right) v_2(t). \quad (2.63)$$

The five conditions at initial and final time for one oscillation with period t_f are

$$x_i(t_f) = x_i(0),\ i=1,2,\ x_2(0)=0,\ v_1(t_f)=v_1(0),\ v_1(0)\,\dot{x}_1(0) + v_2(0)\,\dot{x}_2(0) = 1. \quad (2.64)$$

The period (final time) t_f is free, i. e., determined by the nonlinear boundary condition.

The design parameters β and R_0 have to be determined in a way that the noise

$$J[\beta, R_0] = \frac{\omega^2}{(2\pi f_m)^2}\frac{1}{t_f}\int_0^{t_f} v^T(t)\,G(t)\,\Gamma(t)\,G^T(t)\,v(t)\mathrm{d}t \quad (2.65)$$

is minimized. Here, we consider

$$\Gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad G(t) = \begin{pmatrix} \sqrt{K(t)} & 0 \\ 0 & \sqrt{K(t)} \end{pmatrix} \quad (2.66)$$

with

$$K(t) = \exp\left(x_1^2(t) + x_2^2(t)\right). \quad (2.67)$$

Therefore the objective is equivalent to

$$J[\beta, R_0] = \frac{\omega^2}{(2\pi f_m)^2}\frac{1}{t_f}\int_0^{t_f} K(t)\left(v_1^2(t) + v_2^2(t)\right)\mathrm{d}t. \quad (2.68)$$

The solution has been given explicitly by Anzill (see [1, 39])

$$\begin{array}{lcllcl} \beta &=& 0, & x_2(t) &=& R_0\,\sin(\omega t), \\ R_0 &=& 1, & v_1(t) &=& \left((\beta R_0/\gamma)\cos(\omega t) - \sin(\omega t)\right)/(R_0\omega), \\ x_1(t) &=& R_0\,\cos(\omega t), & v_2(t) &=& \left((\beta R_0/\gamma)\sin(\omega t) + \cos(\omega t)\right)/(R_0\omega). \end{array} \quad (2.69)$$

The period is $t_f = 2\pi/\omega$.

For the computations, we use the following constants

$$f_m = 1, \quad \omega = 2\pi, \quad \gamma = 1. \quad (2.70)$$

In this case, the minimum objective value is $J = \mathrm{e}^1/(2\pi)^2 = 0.068854883...$, and the final time is $t_f = 1$.

2.6.2 `oscillator`: the standard formulation

The problem has one phase, i. e., $m_1 = 1$, $[E_1, E_2] = [0, t_f]$.

The state variables and control parameters are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ \int_0^t K(\tau)\,\left(v_1^2(\tau) + v_2^2(\tau)\right)\,\mathrm{d}\tau \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \beta \\ R_0 \end{pmatrix}. \quad (2.71)$$

New!

There are no control variables, i. e., $l_U = 0$.

The objective according to Section 1.2.1 is

$$\begin{aligned} \Phi &= \Phi_1 \\ \text{where} \quad \Phi_1 &= \frac{\omega^2}{(2\pi f_m)^2} x_5(t_f)/t_f . \end{aligned} \tag{2.72}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \\ f_5^1 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \\ K(t)\,(x_3^2(t) + x_4^2(t)) \end{pmatrix}, \tag{2.73}$$

where $\dot{x}_1$, $\dot{x}_2$, $\dot{v}_1$, $\dot{v}_2$ have to be replaced by the expressions for their right hand sides.

The conditions at the initial time $t_0 = 0$ and final time t_f consist of explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{aligned} x_2(t_0) &= 0, \\ x_5(t_0) &= 0, \end{aligned} \tag{2.74}$$

explicit conditions of the second kind according to Equation (1.11) of Section 1.2.3.1

$$\begin{aligned} x_1(t_f) &= R_{x,1}^1(.) = x_1(t_0), \\ x_2(t_f) &= R_{x,2}^1(.) = x_2(t_0), \\ x_3(t_f) &= R_{x,3}^1(.) = x_3(t_0), \end{aligned} \tag{2.75}$$

and one implicit condition according to Equation (1.13) of Section 1.2.3.2

$$r_1^1(x(t_0), u(t_0), p, t_0, x(t_f), u(t_f), t_f) = 1 - x_3(0) \cdot f_1^1(x(0), p) - x_4(0) \cdot f_2^1(x(0), p) = 0. \tag{2.76}$$

As the upper and lower bounds of Section 1.2.4.1 we use

$$\begin{array}{lllllllllll} \text{1st phase:} & t \in [E_1, E_2], & -10 & = & x_{1,\min}^1 & \le & x_1(t) & \le & x_{1,\max}^1 & = & +10, \\ & & -10 & = & x_{2,\min}^1 & \le & x_2(t) & \le & x_{2,\max}^1 & = & +10, \\ & & -10 & = & x_{3,\min}^1 & \le & x_3(t) & \le & x_{3,\max}^1 & = & +10, \\ & & -10 & = & x_{4,\min}^1 & \le & x_4(t) & \le & x_{4,\max}^1 & = & +10, \\ & & 0 & = & x_{5,\min}^1 & \le & x_5(t) & \le & x_{5,\max}^1 & = & +10^{10}, \end{array} \tag{2.77}$$

and for the control parameters and event

$$\begin{array}{rcccccccl} -10^{-2} & = & p_{1,\min} & \le & p_1 & \le & p_{1,\max} & = & +10^{-2}, \\ 10^{-5} & = & p_{2,\min} & \le & p_2 & \le & p_{2,\max} & = & +10^{10}, \\ 10^{-3} & = & E_{2,\min} & \le & E_2 & \le & E_{2,\max} & = & +10^{10}. \end{array} \tag{2.78}$$

Nonlinear inequality or equality constraints as described in Sections 1.2.4 and 1.2.5 do not appear in this problem. Therefore the dimensions according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
5	0	2	1	0	0	1

2.7 Optimal path tracking for a simple robot

This problem has been suggested to the author by W. Miksch and has been described in Section 6.4 of [39]. The use of ***DIRCOL*** for optimal control of a real mobile manipulator with five degrees of freedom is described in [29].

2.7.1 Problem description

The endeffector of a robot with two rotational joints and simplified equations of motion has to move along a prescribed path with constant velocity. In each joint, there is a motor which can be controlled. The prescribed reference path includes a nonsmooth corner where the reference velocity is discontinuous. Therefore, an error must occur when the robot tries to track this path with constant, non-zero velocity. The overall tracking error has to be minimized.

The second order equations of motion and the initial and final values for position and velocity of the robot are

$$\begin{array}{llllllllllll} \ddot{q}_1(t) & = & u_1(t), & q_1(0) & = & 0, & \dot{q}_1(0) & = & 0.5, & q_1(2) & = & 0.5, & \dot{q}_1(2) & = & 0, \\ \ddot{q}_2(t) & = & u_2(t), & q_2(0) & = & 0, & \dot{q}_2(0) & = & 0, & q_2(2) & = & 0.5, & \dot{q}_2(2) & = & 0.5. \end{array} \tag{2.79}$$

The objective to be minimized is the sum of errors in the tracking of the reference path

$$J[u] := \int_0^2 L(q(t),\dot{q}(t))\,\mathrm{d}t := \int_0^2 \sum_{i=1}^{2} \omega_i (q_i(t) - q_{i,\mathrm{ref}}(t))^2 + \sum_{i=1}^{2} \omega_{2+i} (\dot{q}_i(t) - \dot{q}_{i,\mathrm{ref}}(t))^2\,\mathrm{d}t \tag{2.80}$$

with constant, non-negative, real weighting factors ω_i, $i = 1,2,3,4$. Here, we choose $\omega_1 = \omega_2 = 100$ and $\omega_3 = \omega_4 = 500$. As reference path we use

$$\begin{array}{lll} q_{1,\mathrm{ref}}(t) & = & \begin{cases} t/2, & 0 \le t \le 1, \\ 1/2, & 1 < t \le 2, \end{cases} \quad q_{2,\mathrm{ref}}(t) \;=\; \begin{cases} 0, & 0 \le t \le 1, \\ (t-1)/2, & 1 < t \le 2, \end{cases} \\ \dot{q}_{1,\mathrm{ref}}(t) & = & \begin{cases} 1/2, & 0 \le t \le 1, \\ 0, & 1 < t \le 2, \end{cases} \quad \dot{q}_{2,\mathrm{ref}}(t) \;=\; \begin{cases} 0, & 0 \le t \le 1, \\ 1/2, & 1 < t \le 2. \end{cases} \end{array} \tag{2.81}$$

2.7.2 `robot_i2`: the standard formulation

The problem consists of a single phase with a fixed final time: $m_1 = 1$, $[E_1, E_2] = [t_0, t_f] = [0, 2]$.

The state and control variables are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \\ \int_0^t L(q(\tau),\dot{q}(\tau))\,\mathrm{d}\tau \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{2.82}$$

The objective according to Section 1.2.1 is

New!!

$$\begin{array}{rrcl} & \Phi & = & \Phi_1 \\ \text{where} & \Phi_1 & = & x_5(t_f). \end{array} \tag{2.83}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \\ f_5^1 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ u_1 \\ u_2 \\ L(x(t)) \end{pmatrix}. \tag{2.84}$$

The conditions at the initial time $t_0 = 0$ are explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{array}{rcl} x_1(t_0) & = & 0, \\ x_2(t_0) & = & 0, \\ x_3(t_0) & = & 0.5, \\ x_4(t_0) & = & 0, \\ x_5(t_0) & = & 0. \end{array} \tag{2.85}$$

The conditions at the final time $t_f = 1$ are explicit conditions of the second kind according to Equation (1.11) of Section 1.2.3.1

$$\begin{array}{rcccl} x_1(t_f) & = & R^1_{x,1}(.) & = & 0.5, \\ x_2(t_f) & = & R^1_{x,2}(.) & = & 0.5, \\ x_3(t_f) & = & R^1_{x,3}(.) & = & 0, \\ x_4(t_f) & = & R^1_{x,4}(.) & = & 0.5, \\ x_5(t_f) & \text{is} & \text{free.} & & \end{array} \tag{2.86}$$

The state variables are unconstrained. For the control variables we use a bound of 10 for a maximum acceleration and deceleration. Therefore, as the upper and lower bounds of Section 1.2.4.1 one obtains in the

$$\text{1st phase:} \quad t \in [E_1, E_2], \quad \begin{array}{rcccccccl} -10^{10} & = & x^1_{1,\min} & \leq & x_1(t) & \leq & x^1_{1,\max} & = & +10^{10}, \\ -10^{10} & = & x^1_{2,\min} & \leq & x_2(t) & \leq & x^1_{2,\max} & = & +10^{10}, \\ -10^{10} & = & x^1_{3,\min} & \leq & x_3(t) & \leq & x^1_{3,\max} & = & +10^{10}, \\ -10 & = & u^1_{1,\min} & \leq & u_1(t) & \leq & u^1_{1,\max} & = & +10, \\ -10 & = & u^1_{2,\min} & \leq & u_2(t) & \leq & u^1_{2,\max} & = & +10. \end{array} \tag{2.87}$$

Control parameters do not appear in this problem, i. e., $l_P = 0$. The final time is fixed. Therefore, the lower and upper bounds for E_2 are

$$2 = E_{2,\min} \leq E_2 \leq E_{2,\max} = 2. \tag{2.88}$$

Nonlinear inequality or equality constraints as described in Sections 1.2.4 and 1.2.5 do not appear in this problem. Also, there are no implicit boundary conditions of the form of Section 1.2.3.2.

Hence, the dimensions according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
5	2	0	1	0	0	0

2.8 Abort landing in the presence of a windshear

"One of the most dangerous situations for a passenger aircraft in take-off and landing is caused by the presence of low-altitude windshears ... Even for a highly-skilled pilot, an inadvertent encounter with a windshear can be a fatal problem, since the aircraft might encounter a headwind followed by a tailwind, both coupled with a down draft ... the abort landing problem ... is a safer procedure than the penetration landing if the initial altitude is high enough" [8].

The following statement of the problem has been taken from [8], also the reference solution of the min-max problem which is included in the supplied files. The investigation of the problem was highly motivated by the previous work of [27, 28].

2.8.1 Problem description

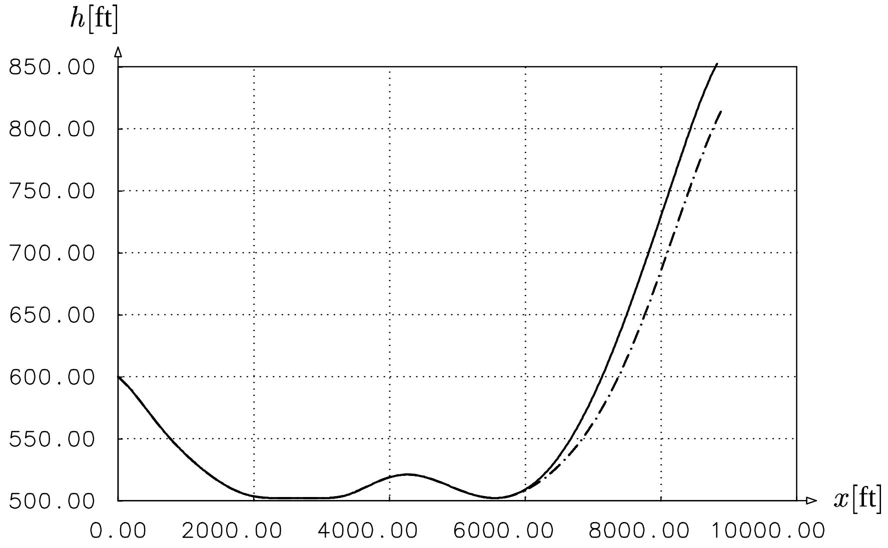


Figure 2.7: History of the altitude $h(t)$ of the passenger aircraft versus range $x(t)$ (computed solutions of ***DIRCOL*** (–·–·–) and multiple shooting (———)) [39].

The five state variables are the horizontal distance x, the altitude h, the relative velocity v, the relative path inclination γ, and the relative angle of attack α. The control variable $u = \dot{\alpha}$ is the time derivative of the angle of attack.

The total time of the flight manoeuvre considered is $t_f = 40$ s.

The given values of the state variables at initial and final time are

$$\begin{array}{rcrlcl} x(0) & = & 0[\text{ft}], & x(t_f) & & \text{is free,} \\ h(0) & = & 600[\text{ft}], & h(t_f) & & \text{is free,} \\ v(0) & = & 239.7[\text{ft/s}], & v(t_f) & & \text{is free,} \\ \gamma(0) & = & -0.03925[\text{rad}], & \gamma(t_f) & = & 0.12969[\text{rad}], \\ \alpha(0) & = & 0.1283[\text{rad}], & \alpha(t_f) & & \text{is free.} \end{array} \qquad (2.89)$$

The objective to be maximized is the minimum altitude attained during the flight manoeuvre

$$J[u] \;=\; \min_{0 \le t \le t_f} h(t) \;\longrightarrow\; \max! \qquad (2.90)$$

The equations of motion of the aircraft are

$$\begin{aligned}
\dot{x} &= v\cos\gamma + W_1(x),\\
\dot{h} &= v\sin\gamma + W_2(x,h),\\
\dot{v} &= \frac{T(v)}{m}\cos(\alpha+\delta) - \frac{D(v,\alpha)}{m} - g\sin\gamma - \left(\dot{W}_1(x)\cos\gamma + \dot{W}_2(x,h)\sin\gamma\right),\\
\dot{\gamma} &= \frac{T(v)}{mv}\sin(\alpha+\delta) + \frac{L(v,\alpha)}{mv} - \frac{g}{v}\cos\gamma + \frac{1}{v}\left(\dot{W}_1(x)\sin\gamma - \dot{W}_2(x,h)\cos\gamma\right),\\
\dot{\alpha} &= u\,.
\end{aligned} \tag{2.91}$$

The thrust T is given by

$$T(v) = \beta(t)(A_0 + A_1 v + A_2 v^2) \quad \text{with} \quad \beta(t) = \begin{cases} \beta_0 + \dot{\beta}_0 t, & 0 \le t \le t_\beta,\\ 1, & t_\beta \le t \le t_f, \end{cases} \tag{2.92}$$

and

$$\beta_0 = 0.3825, \quad \dot{\beta}_0 = 0.2[\mathrm{s}^{-1}] \quad \text{and} \quad t_\beta = (1-\beta_0)/\dot{\beta}_0 = 3.0875[\mathrm{s}].$$

For the drag D holds

$$\begin{aligned}
D(v,\alpha) &= 0.5\, C_D(\alpha)\,\rho\, S\, v^2 \\
\text{where}\quad C_D(\alpha) &= B_0 + B_1\alpha + B_2\alpha^2,\\
\rho &= 0.2203\cdot 10^{-2}[\mathrm{lb\,s^2 ft^{-4}}], \quad S = 0.1560\cdot 10^4[\mathrm{ft}^2].
\end{aligned} \tag{2.93}$$

The lift L is described by

$$\begin{aligned}
L(v,\alpha) &= 0.5\, C_L(\alpha)\,\rho\, S\, v^2 \\
\text{where}\quad C_L(\alpha) &= \begin{cases} C_0 + C_1\alpha, & \alpha \le \alpha_\star,\\ C_0 + C_1\alpha + C_2(\alpha-\alpha_\star)^2, & \alpha_\star < \alpha \le \alpha_{\max}, \end{cases}\\
\alpha_\star &= 12[\mathrm{deg}] \doteq 0.20943951[\mathrm{rad}], \quad \alpha_{\max} = 0.3002[\mathrm{rad}].
\end{aligned} \tag{2.94}$$

The windshear model, valid for $h \le 1000[\mathrm{ft}]$, is given by

$$W_1(x) = \begin{cases} -50 + ax^3 + bx^4, & 0 \le x < 500,\\ (x-2300)/40, & 500 \le x < 4100,\\ 50 - a(4600-x)^3 - b(4600-x)^4, & 4100 \le x \le 4600,\\ 50, & 4600 \le x, \end{cases} \tag{2.95}$$

$$W_2(x,h) = \frac{h}{1000}\cdot \begin{cases} (dx^3 + ex^4)\,, & 0 \le x < 500,\\ (-51\exp(-c(x-2300)^4))\,, & 500 \le x < 4100,\\ (d(4600-x)^3 + e(4600-x)^4)\,, & 4100 \le x \le 4600,\\ 0, & 4600 < x. \end{cases} \tag{2.96}$$

The constants are (cf. [8, 30])

$$\begin{array}{lll lll}
A_0 &=& +0.4456\cdot 10^5[\mathrm{lb}], & B_0 &=& +0.15523333333,\\
A_1 &=& -0.2398\cdot 10^2[\mathrm{lb\,s\,ft^{-1}}], & B_1 &=& +0.1236914764[\mathrm{rad}^{-1}],\\
A_2 &=& +0.1442\cdot 10^{-1}[\mathrm{lb\,s^2\,ft^{-2}}], & B_2 &=& +2.420265075[\mathrm{rad}^{-2}],\\
C_0 &=& +0.7125, & a &=& +6\cdot 10^{-8}[\mathrm{s^{-1}ft^{-2}}],\\
C_1 &=& +6.087676573[\mathrm{rad}^{-1}], & b &=& -4\cdot 10^{-11}[\mathrm{s^{-1}ft^{-3}}],\\
C_2 &=& -9.027717451[\mathrm{rad}^{-2}], & c &=& -\ln(25/30.6)\cdot 10^{-12}[\mathrm{ft}^{-4}],\\
d &=& -8.028808625\cdot 10^{-8}[\mathrm{s^{-1}ft^{-2}}], & e &=& +6.280834899\cdot 10^{-11}[\mathrm{s^{-1}ft^{-3}}],\\
g &=& +3.2172\cdot 10^1[\mathrm{ft\,s^{-2}}], & mg &=& 150000[\mathrm{lb}],\\
\delta &=& 2[\mathrm{deg}] \doteq 0.034906585[\mathrm{rad}]. &&&
\end{array} \tag{2.97}$$

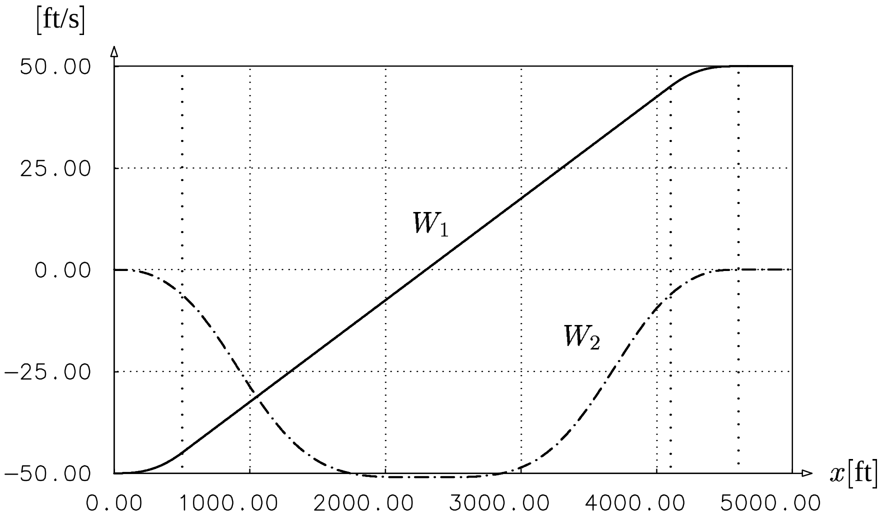


Figure 2.8: Horizontal and vertical components of the wind velocity.

The angle of attack and its time derivative are constrained by

$$|\alpha(t)| \leq \alpha_{\max} = 0.3002[\text{rad}], \tag{2.98}$$

$$|u(t)| \leq u_{\max} = 0.05236[\text{rad}]. \tag{2.99}$$

In [8, 30]

$$p^* = \min\{h^*(t) : t \in [0, t_f]\} = 502.15627829[\text{ft}] \tag{2.100}$$

has been computed for the objective.

2.8.2 `windshear`: the standard formulation

The problem has one phase, i. e., $m_1 = 1$, $[E_1, E_2] = [0, t_f] = [0, 40]$.

For transformation of the min-max problem into standard form with a Mayer-type objective one control parameter p_1 for the minimal altitude is introduced by (cf. Section 1.3.3)

$$p_1 := \min_{0 \le t \le t_f} h(t). \tag{2.101}$$

Then, an additional state constraint has to be taken into account

$$g_1^1(x, u, p, t) = x_2(t) - p_1 = h(t) - p_1 \ge 0, \quad 0 \le t \le t_f. \tag{2.102}$$

The transformed objective to be minimized becomes

$$\tilde{J}[u, p] = -p_1. \tag{2.103}$$

The state variables, control variables, and control parameters are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x \\ h \\ v \\ \gamma \\ \alpha \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \end{pmatrix}. \tag{2.104}$$

New!!

The objective according to Section 1.2.1 is

$$\begin{aligned} \Phi &= \Phi_1 \\ \text{where} \quad \Phi_1 &= -p_1. \end{aligned} \tag{2.105}$$

According to Section 1.2.2, the right hand side of the differential equations reads as

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \\ f_5^1 \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{h} \\ \dot{v} \\ \dot{\gamma} \\ u_1 \end{pmatrix}, \tag{2.106}$$

where $\dot{x}$, $\dot{h}$, $\dot{v}$, $\dot{\gamma}$ have to be replaced by the expressions for their right hand sides.

The conditions at the initial time $t_0 = 0$ and final time $t_f = 40$ consist of explicit conditions of the first kind according to Equation (1.10) of Section 1.2.3.1

$$\begin{aligned} x_1(t_0) &= 0, \\ x_2(t_0) &= 600, \\ x_3(t_0) &= 239.7, \\ x_4(t_0) &= -0.03925, \\ x_5(t_0) &= 0.1283, \end{aligned} \tag{2.107}$$

and one explicit condition of the second kind according to Equation (1.11) of Section 1.2.3.1

$$x_4(t_f) = R_{x,4}^1(.) = 7.431 \cdot \pi/180 \doteq 0.1296942. \tag{2.108}$$

As the upper and lower bounds for the state and control variables of Section 1.2.4.1 we use

$$\text{1st phase:} \quad t \in [E_1, E_2], \quad \begin{array}{rcccccccl} 0 &=& x^1_{1,\min} &\leq& x_1(t) &\leq& x^1_{1,\max} &=& +1.5 \cdot 10^4, \\ +10^2 &=& x^1_{2,\min} &\leq& x_2(t) &\leq& x^1_{2,\max} &=& +1.5 \cdot 10^3, \\ 0 &=& x^1_{3,\min} &\leq& x_3(t) &\leq& x^1_{3,\max} &=& +10^{10}, \\ -10^{10} &=& x^1_{4,\min} &\leq& x_4(t) &\leq& x^1_{4,\max} &=& +10^{10}, \\ -\alpha_{\max} &=& x^1_{5,\min} &\leq& x_5(t) &\leq& x^1_{5,\max} &=& +\alpha_{\max}, \\ -u_{\max} &=& u^1_{1,\min} &\leq& u_1(t) &\leq& u^1_{1,\max} &=& +u_{\max}. \end{array} \tag{2.109}$$

The state variable $x_4 = \gamma$ is an unconstrained angle but varies only within $[-\pi, +\pi[$. Therefore, the option for special treatment of angles is selected for x_4 in the input file DATDIM (cf. Section 4.2.1.1).

The lower and upper bounds for the control parameters and events are

$$\begin{array}{rcccccccl} +10 &=& p_{1,\min} &\leq& p_1 &\leq& p_{1,\max} &=& +10^{10}, \\ +40 &=& E_{2,\min} &\leq& E_2 &\leq& E_{2,\max} &=& +40. \end{array} \tag{2.110}$$

The inequality constraint of Equation (2.102) is formulated according to Section 1.2.4

$$\text{1st phase:} \quad t \in [E_1, E_2], \quad g^1_1(x, u, p, t) \;=\; x_2(t) - p_1 \;\geq\; 0. \tag{2.111}$$

Nonlinear equality constraints as described in Section 1.2.5 do not appear in this problem. Therefore, the dimensions of this optimal control problem according to Section 1.2 are

n_X	l_U	l_P	m_1	$n_{g,nln,1}$	$n_{h,nln,1}$	$n_{r,nln,1}$
5	1	1	1	1	0	0

2.9 Optimal point-to-point trajectories of a 3-d.o.f. industrial robot

New!!

2.9.1 Minimum time trajectory: problem description

see [39, 41]

The subroutine R3M2SI is due to Otter and Türk [31]. It is included by friendly permission of Dr.-Ing. Martin Otter [32].

2.9.2 `manutec_minimum_time`: the standard formulation

2.9.3 Minimum energy trajectory: problem description

2.9.4 `manutec_minimum_energy`: the standard formulation

2.10 References to other successful applications of *DIRCOL*

In this section, a selection of references to successful applications of ***DIRCOL*** is given.

An optimal control problem in economics with four linear controls describing a sophisticated concern model is investigated by Koslik and Breitner [21]. The numerical solution is obtained by a combination of the direct collocation method ***DIRCOL*** and an indirect multiple shooting method for solving the necessary conditions from optimal control theory. The optimal controls have bang-bang subarcs as well as constrained and singular subarcs. Here, the highly complicated switching structure has been computed correctly using ***DIRCOL***. Also, the computed estimates of adjoint variables and the computed histories of the switching functions have been estimated in an accuracy of four to five decimals. Thus, the subsequent solution of the necessary conditions from optimal control theory, i. e., the multi-point boundary value problem for the state and adjoint variables, has been facilitated enormously.

Zoelch [44, 45] applies ***DIRCOL*** successfully for optimizing and rating of the design of a hybrid driveline.

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