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Optimal Control of Multibody Systems in Minimal Coordinates

The present paper deals with numerical methods for optimal control of mechanical multibody systems, such as many industrial robots, whose dynamical behavior can be described in minimal coordinates by a system of semi-explicit second order differential equations: $M(q(t))\,\ddot{q}(t) = u(t) + h(q(t), \dot{q}(t), t)$, $0 \le t \le t_f$, where q denotes the state variables, u the control variables, and $M(q)$ the positive definite mass matrix. Numerical methods are addressed for computing an approximation of the optimal control $u^*(t)$, $0 \le t \le t_f$, which steers the system from an initial to a final position minimizing a performance index, such as time or energy, subject to bounds or nonlinear constraints on q, $\dot{q}$, and u. A method is investigated in more detail which is based on piecewise polynomial approximations of state variables and utilizes the structure of the dynamical equations as well as the structure in the resulting large and sparse, nonlinearly constrained optimization problems. Results for an industrial robot with six joints demonstrate that tailored optimization methods are very well suited for fast off-line optimization of robot trajectories.

1. Introduction

In automotive industry, robotic manipulators play an important role for car production in automated assembly lines. The reduction of the cycle time of the production line is of great importance in order to reduce costs by improving efficiency. Here “optimal” robot trajectories and robust and efficient optimization methods for computing them are of great interest when planning production processes.

2. Modeling of the optimization problem

EQUATIONS OF MOTION. Industrial robots have to act extremely fast. Therefore all dynamical effects have to be taken into account. The dynamical behavior of most industrial robots can be described in minimal coordinates by a system of second order differential equations (multibody system with tree structure [18]). The *simulation model*

$$\begin{pmatrix} M_{1,1}(q(t)) & \dots & M_{1,n}(q(t)) \\ \vdots & & \vdots \\ M_{n,1}(q(t)) & \dots & M_{n,n}(q(t)) \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \vdots \\ \ddot{q}_n(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} + \begin{pmatrix} h_1\,(q(t), \dot{q}(t), t) \\ \vdots \\ h_n\,(q(t), \dot{q}(t), t) \end{pmatrix} \tag{1}$$

is used for computing the generalized accelerations $\ddot{q}(t)$ for given $q(t)$, $\dot{q}(t)$, $u(t)$, $0 \le t \le t_f$. Here n denotes the number of joints, $q = (q_1, \dots, q_n)^T$ the state variables, $u = (u_1, \dots, u_n)^T$ the control variables, $M = M(q) \in \mathbb{R}^{n \times n}$ the positive definite and symmetric mass matrix, and $h = h(q, \dot{q}, t)$ the moments resulting from gravitational, centrifugal, Coriolis, and frictional forces. Eq. (1) can easily be solved for the control variable yielding the *inverse model*

$$u(t) \;=\; M(q(t))\,\ddot{q}(t) - h\,(q(t), \dot{q}(t), t)\;, \tag{2}$$

which is used for computing the generalized joint forces $u(t)$ needed for a given trajectory $q(t)$, $\dot{q}(t)$, $\ddot{q}(t)$, $0 \le t \le t_f$.
OBJECTIVES. We investigate *point-to-point* trajectories from a given initial position A to a final position B:

$$\text{A:} \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \qquad \text{B:} \quad q(t_f) = q_f, \quad \dot{q}(t_f) = \dot{q}_f, \tag{3}$$

with given constants q_0, $\dot{q}_0$, q_f, $\dot{q}_f \in \mathbb{R}^n$. Usually stationary boundary conditions are considered, i. e., $\dot{q}_0 = \dot{q}_f = 0$. The control variable $u(t)$ is choosen in order to minimize a performance index J

$$J[u] = \phi\,(q(t_f), \dot{q}(t_f), t_f) + \int_0^{t_f} L\,(q(t), \dot{q}(t), u(t), t)\;\mathrm{d}t \qquad (\phi : \mathbb{R}^{2n+1} \to \mathbb{R},\; L : \mathbb{R}^{3n+1} \to \mathbb{R}) \tag{4}$$

such as *time*: $J_\mathrm{t}[u] = t_f$, or *energy*: $J_\mathrm{e}[u] = \int_0^{t_f} \sum_{i=1}^n u_i^2(t)\,\mathrm{d}t$, the first with a free and the latter with a prescribed final time t_f.
CONSTRAINTS. Several technical and geometrical constraints on the robot trajectories have to be taken into account. First of all, the torque controls are limited by given constants $u_i^{\min}$ and $u_i^{\max}$

$$u_i^{\min} \;\le\; u_i(t) \;\le\; u_i^{\max}, \quad i = 1, \dots, n\,. \tag{5}$$

In general, the control and state variables have to satisfy nonlinear inequality constraints of the form

$$c(q(t), \dot{q}(t), u(t), t) \leq 0 \tag{6}$$

which includes geometrical constraints in order to avoid collisions within the working cell. Especially box constraints on the state variables and their velocities often have to be considered for given bounds $q_i^{\min}, q_i^{\max}, \dot{q}_i^{\min}, \dot{q}_i^{\max} \in \mathbb{R}$

$$q_i^{\min} \leq q_i(t) \leq q_i^{\max}, \quad \dot{q}_i^{\min} \leq \dot{q}_i(t) \leq \dot{q}_i^{\max}, \quad i = 1, \ldots, n\,. \tag{7}$$

3. Methods for trajectory optimization

The robot trajectory optimization problem described in the previous section is an optimal control problem. Two groups of numerical methods can be found in the literature.

1. General Methods. Methods for general optimal control problems can be applied to the robot trajectory problem of Section 2. This requires the transformation of the problem to a first order "standard" form. By using $x = (q_1, \ldots, q_n, \dot{q}_1, \ldots \dot{q}_n)^T \in \mathbb{R}^{2n}$ the problem reads as

$$\begin{array}{lrcl} \text{minimize} & J[u] & = & \phi(x(t_f), t_f) + \int_0^{t_f} L\,(x(t), u(t), t)\,\mathrm{d}t \\ \text{subject to} & \dot{x}(t) & = & f(x(t), u(t), t)\,, \quad 0 \leq t \leq t_f\,, \\ \text{and} & 0 & \geq & c(x(t), u(t), t)\,, \\ & 0 & = & r(x(0), x(t_f), t_f)\,, \end{array} \tag{8}$$

$$\text{where} \quad \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{2n} \end{pmatrix}, \quad \begin{pmatrix} \dot{x}_{n+1} \\ \vdots \\ \dot{x}_{2n} \end{pmatrix} = M^{-1}(x) \cdot (u + h(x,t))\,. \tag{9}$$

Numerical Methods for solving problem (8) can be classified into *direct* and *indirect* methods [22]. Indirect methods, such as the multiple shooting method, gradient and min-H methods, are mainly based on the Maximum Principle. Here the adjoint variables $\lambda(t)$, the adjoint equations $\dot{\lambda} = -\lambda^T\, \partial f / \partial x$ etc. are used explicitly. Indirect methods are often cumbersome to apply as they require to deal with optimal control theory (e. g., adjoint variables).
On the other hand direct (transcription) methods don't have to deal with adjoint variables and necessary conditions from optimal control theory explicitly. They are based on a discretization of control and state variables and the equations of motion, e. g., by piecewise polynomial approximations $\tilde{u}(t) = \sum_{i=1}^{N-1} \alpha_i\, \hat{u}_{(i)}(t)$, $\alpha_i \in \mathbb{R}^n$, and $\tilde{x}(t) = \sum_{i=1}^{N-1} \beta_i\, \hat{x}_{(i)}(t)$, $\beta_i \in \mathbb{R}^{2n}$ [1,5,21]. The resulting large, finite-dimensional, nonlinearly constrained optimization problems (NLPs) can be solved efficiently by Sequential Quadratic Programming (SQP) methods [9,19]. Most common are direct shooting [1,2,15] and direct collocation methods [10,21]. A combination of direct collocation and indirect multiple shooting methods which combines the merits of both has been developed in [21,22,23]. It has been demonstrated that carefully implemented direct methods do provide reliable, easy-to-use and robust tools for solving optimal control problems if accuracy requirements are not extremely high. For direct transcription methods typically the resulting NLPs are large and sparse and grid refinement is required in order to adapt the discretization.

2. Tailored Methods. Several methods have been developed which are tailored to the robot trajectory optimization problem (1) – (7) but are not applicable to general optimal control problems (8). E. g., one widespread tailored method described in [13] consists of two parts: (a) the optimization of the velocity profile along a given geometrical path and (b) the optimization of the geometrical path itself.
In [6] a discretization of problem (1) – (7) has been suggested based on a parameterization of q by equidistant cubic splines and using the inverse dynamic model of Eq. (2). This yields an explicit formula for an approximation of the control $\tilde{u}$ and formally eliminates the dynamic equations a priori and therefore the need for numerical integration. A similar idea has previously been suggested in [20] as state parameterization approach for systems where the dimension of the control variables equals the dimension of the state variables (as in Eq. (9)).
Recently, a robot trajectory optimization method based on a state parameterization of Eq. (2) by piecewise continuous cubic splines and collocation at Gaussian points has been developed in [11]. This approach satisfies the dynamic equations only at discrete points by a set of nonlinear equality constraints for the resulting NLP. It has been demonstrated that this discretization avoids the well-known oscillatory effects of solutions from direct transcription methods in the case of active state constraints. This is mainly due to the fact that the chosen approximation of $\ddot{\tilde{q}}$ may be discontinuous at the matching points of the discretization intervals. The state variable often is continuous but not differentiable when the optimal trajectory enters or leaves an active state constraint. This method has demonstrated to be robust and efficient and can conveniently and successfully be used with computer-aided production engineering tools in automotive industry (see [11] for details).

4. A direct transcription method utilizing structure

State Parameterization. We use the state parameterization suggested in [6] in order to take into account the structure of the equations of motion (1), (2). The state variable q is approximated by piecewise cubic splines, i. e.,

$$\tilde{q}_i(t) \;=\; \sum_k y_{k,i} \cdot N_{k,4}(t)\,, \quad y_{k,i} \in \mathbb{R}\,, \quad i = 1, \ldots, n\,, \tag{10}$$

where $N_{k,4}$ is the k-th normalized cubic B-spline on a suitable time grid [3]. Here we choose an equidistant grid and an overall continuously differentiable approximation in $[0, t_f]$. By differentiating (10) with respect to time we obtain a piecewise quadratic approximation of $\dot{\tilde{q}}$, which is continuous in $[0, t_f]$, and a piecewise linear approximation of $\ddot{\tilde{q}}$. Then, a piecewise nonlinear approximation of $\tilde{u}$ is obtained using the inverse model of Eq. (2)

$$\tilde{u}(t) \;=\; M(\tilde{q}(t)) \cdot \ddot{\tilde{q}}(t) - h\left(\tilde{q}(t), \dot{\tilde{q}}(t), t\right)\,. \tag{11}$$

This approximation of u depends on the parameters $y_{k,i}$ of the state parameterization $\tilde{q}$ and on the inverse model only. Compared to standard direct transcription methods [1,10,21] the collocation equations can be eliminated a priori from the nonlinear equality constraints of the NLP of the discretized problem.

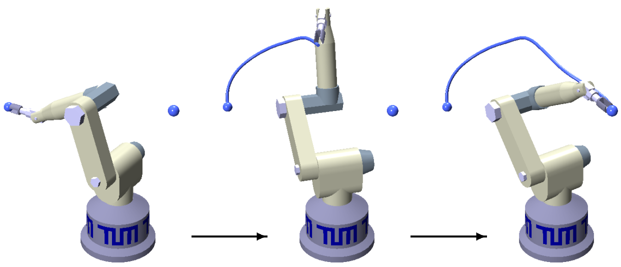
Discretization of Constraints. The box constraints on q and $\dot{q}$ of Eq. (7) become linear constraints on $y_{k,i}$ in the discretized problem. The box constraints on u of Eq. (5) become *nonlinear* inequality constraints on $y_{k,i}$ in the discretized problem. In standard direct transcription methods [1,2,5,10,15,21,22,23] with a straight forward discretization of u, e. g., by piecewise linear polynomials, the box constraints on u become box constraints also on the parameters of the discretization. Box constraints are generally easier to handle than nonlinear inequality constraints in the numerical solution of NLPs. In minimum time trajectories the box constraints on u and on $\dot{q}$ become active very often during the motion. This is the price to pay for utilizing dynamical structure by the proposed state parameterization. The gains are smaller dimensional NLPs of the discretized problems.

Computation of Gradients. Gradients of objective and constraint functions of the NLP of the discretized problem are computed by finite difference approximations [8]. Reliable and efficient gradients of the Langrangian part $\int_0^{t_f} L(.)\,\mathrm{d}t$ of the objective (4) are computed by inital value problem methods for sensitivity matrices [4].

Sparse Nonlinear Optimization. The parameters of the discretized robot trajectory optimization problem (1) – (7) are the coefficients $y_{k,i}$ of the state parameterization (10) and the possibly free final time t_f. The objective of the NLP is the discretized objective (4). The constraints of the NLP are the discretized constraints (3), (5), (6), (7) depending on the parameters of the discretization. They have to be fulfilled at the time points of the discretization grid. The NLP can be solved by state-of-the-art general-purpose SQP methods such as NPSOL [9]. On the other hand the number of degrees of freedom of the NLP is significantly smaller than the number of variables as many constraints are active at the solution. Therefore a “sparse” SQP method which utilizes NLP structure will generally be more efficient. Here we use the recently developed algorithm SNOPT which uses a limited memory quasi-Newton approximation of the Hessian of the NLP-Lagrangian and a reduced Hessian algorithm for the QP subproblems [7].

5. Results

We consider an industrial robot of type Manutec r3 with $n = 6$ revolute joints. Simulation and inverse model of the equations of motion are given as subroutines in [17]. The minimum time point-to-point motion described in Sec. 6.3.3 of [14] and Sec. 8.5.3 of [21] and given by A: $q(0) = (13.1, 17.8, 100.5, -1.6, -28.3, 18.9)^T$[deg], B: $q(t_f) = (-8.4, 41.9, -104.8, -1.5, -35.8, -42.4)^T$[deg] is considered (see figure below). The box constraints on q and $\dot{q}$ of Eq. (7) are taken into account with the upper and lower limits given in [17]. The results for the general-purpose direct transcription method DIRCOL with NPSOL (S) [21], the state parameterization proposed in Eqs. (10) and (11) with NPSOL (T1) and SNOPT (T2) are listed below for 11 and 51 equidistant grid points (see [16] for details). The CPU-seconds refer to a Sun Sparc station 10. A Linux-PC with a Pentium-Pro 200 MHz is four times faster.



n_{grid}		$n_{\text{var.}}$	$n_{\text{nl.eqs.}}$	$n_{\text{nl.ineqs.}}$	Jacobian	$n_{\text{itns.}}$	CPU [s]	t_f [s]
11	S:	175	120	0	20.1 %	25	148.	1.0893
	T1:	79	—	144	23.4 %	49	46.	1.0877
	T2:	79	—	144	23.4 %	45	16.	1.0878
51	S:	895	600	0	4.1 %	24	9565.	1.0812
	T1:	319	—	624	5.9 %	51	2176.	1.0812
	T2:	319	—	624	5.9 %	95	282.	1.0812

6. Conclusions

The trajectory optimization problem for industrial robots can be described in mathematical terms as an optimal control problem. Its solution is the "best" trajectory with respect to the considered performance index. Efficient numerical methods have been described which are well suited for fast off-line optimization of robot trajectories.

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