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Optimal Control of the Industrial Robot Manutec r3

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Abstract

Minimum time and minimum energy point-to-point trajectories for an industrial robot of the type Manutec r3 are computed subject to state constraints on the angular velocities. The numerical solutions of these optimal control problems are obtained in an efficient way by a combination of a direct collocation and an indirect multiple shooting method. This combination links the benefits of both approaches: A large domain of convergence and a highly accurate solution. The numerical results show that the constraints on the angular velocities become active during large parts of the time optimal motion. But the resulting stress on the links can be significantly reduced by a minimum energy trajectory that is only about ten percent slower than the minimum time trajectory. As a by-product, the reliability of the direct collocation method in estimating adjoint variables and the efficiency of the combination of direct collocation and multiple shooting is demonstrated. The highly accurate solutions reported in this paper may also serve as benchmark problems for other methods.

1 Introduction

With the increasing use of robotic manipulators the requirements of their abilities are also increasing. An essential part in design and application of robots is their dynamic behaviour. The discussion of optimal trajectories within the context of path planning and optimal design of parameters leads to the optimal control problems discussed in this paper.

Several methods for solving optimal point-to-point trajectory problems of robotic manipulators have been suggested and applied, e. g., in [7], [13], [14], [15], [23], to cite only a few of many papers.

As an extension to the previous cited work we investigate a non academic, highly nonlinear model of a commercially available robot, discuss several objectives for optimal trajectories and consider state constraints on the angular velocities that play an important role in the time optimal motion.

In our approach, we combine a direct collocation and an indirect multiple shooting method in an hybrid approach (cf. [28]) with a large domain of convergence and highly accurate solutions. The direct collocation method is easily capable to treat a wide variety of objectives and constraints on the state and control variables.

2 Problem Statement

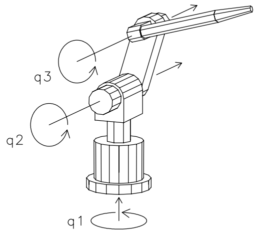

Figure 1: Three degrees of freedom in the DLR model 2 of the r3 robot.

We consider the Manutec r3 robot with 6 links. As the first 3 degrees of freedom (d.o.f.) are mainly responsible for the position and the last 3 d.o.f. for the orientation of the tool center point frame, we restrict ourself to the (first) 3 d.o.f. case. The DLR model 2 of the Manutec r3 robot was developed by Otter and Türk [19] and describes the motion of the links as a function of the control input signals of the robot drives

$$M(q(t)) \cdot \ddot{q}(t) \;=\; D \cdot u(t) + \chi^d(\dot{q}(t), q(t)) + \chi^g(q(t)), \quad t \in [0, t_f], \tag{1}$$

where $q = (q_1(t), q_2(t), q_3(t))^T$ are the relative angles between the arms,the normalized torque controls are $u = (u_1(t), u_2(t), u_3(t))^T$, $D = \mathrm{diag}(d_1, d_2, d_3)$ is a scaling matrix with $d_i = \mathrm{const}[\mathrm{Nm/V}]$, $M(q)$ is the positive definite and symmetric (3×3)-matrix of moments of inertia, $\chi^d(\dot{q}(t), q(t))$ are the moments caused by coriolis and centrifugal forces, and $\chi^g(q(t))$ are the moments caused by gravitational forces. The final time t_f may be prescribed or free. The full data of the dynamic model can be found in [19]. Just to give an impression of the model we give the structure of the first element of the mass matrix M

$$\begin{aligned} M_{1,1}(q) \;&=\; c_1\,(\sin(q_2+q_3))^2 + c_2 \sin(q_2+q_3)\sin(q_2) \\ &\quad + c_3\,(\sin(q_2))^2 + c_4\,(\cos(q_2+q_3))^2 + c_5\,(\cos(q_2))^2 + c_6, \end{aligned} \tag{2}$$

where $c_i = \mathrm{const}$, $i = 1, \ldots, 6$, and of the driving forces $\chi^d(\dot{q}, q)$

$$\chi_i^d(\dot{q}, q) \;=\; \sum_{j=1}^{3} \left(\sum_{k=1}^{3} \Gamma_{i,j\,k}(q)\ \dot{q}_k \right) \dot{q}_j, \quad i = 1, 2, 3, \tag{3}$$

$$\text{with} \quad \Gamma_{i,j\,k}(q) \;=\; -\frac{1}{2} \left(\frac{\partial M_{i,j}(q)}{\partial q_k} + \frac{\partial M_{i,k}(q)}{\partial q_j} - \frac{\partial M_{j,k}(q)}{\partial q_i} \right). \tag{4}$$

The dynamic behaviour of the robot is now given either in an efficient implicit form of the right hand side of $\ddot{q} = M^{-1}(Du + \chi^d + \chi^g)$ by the subroutine R3M2SI [19] or explicitly by the output of a symbolic computation system given in the appendix of [19].

Point-to-point trajectories are to be considered, i. e.,

$$q(0) = q_0, \quad q(t_f) = q_f, \quad \dot{q}(0) = \dot{q}_0, \quad \dot{q}(t_f) = \dot{q}_f. \tag{5}$$

Here, we consider stationary boundary conditions, i. e., $\dot{q}_0 = \dot{q}_f = 0$. As objectives for optimal trajectories three criterions are investigated: The minimum time

$$J_1[u, t_f] \ := \ t_f \ \rightarrow \text{min!} \ , \tag{6}$$

the minimum energy

$$J_2[u] \ := \ \int_0^{t_f} \sum_{i=1}^{3} (u_i(t))^2 \, \mathrm{d}t \ \rightarrow \text{min!} \ , \tag{7}$$

and the minimum power consumption (cf. [16], [21])

$$J_3[u] \ := \ \int_0^{t_f} \sum_{i=1}^{3} (\dot{q}_i(t) u_i(t))^2 \, \mathrm{d}t \ \rightarrow \text{min!} \tag{8}$$

The final time t_f has to be prescribed for J_2 and J_3 in order to obtain useful solutions. Otherwise, a free t_f will tend to become very large. Eighteen technical constraints have to be considered (cf. [19]): There are control constraints on the torque voltages

$$|u_i(t)| \ \leq \ u_{i,\max} \ = \ 7.5[V], \quad i = 1, 2, 3, \tag{9}$$

state constraints on the angles

$$\begin{aligned} |q_1(t)| &\leq 2.97[\text{rad}], \\ |q_2(t)| &\leq 2.01[\text{rad}], \\ |q_3(t)| &\leq 2.86[\text{rad}], \end{aligned} \tag{10}$$

and state constraints on the angular velocities

$$\begin{aligned} |\dot{q}_1(t)| &\leq 3.0[\text{rad/s}], \\ |\dot{q}_2(t)| &\leq 1.5[\text{rad/s}], \\ |\dot{q}_3(t)| &\leq 5.2[\text{rad/s}]. \end{aligned} \tag{11}$$

The numerical results show that the latter constraints become often active during the time optimal motions. Thus they play an important role within the optimization.

3 Numerical Methods

In order to derive the necessary conditions of optimality and to apply the general numerical methods, a formal transformation of the second order system to a first order one has to be performed. First the notation

$$x = (x^1, x^2)^T = (x_1, \ldots, x_6)^T, \quad x^1 := (q_1, q_2, q_3)^T, \quad x^2 := (\dot{q}_1, \dot{q}_2, \dot{q}_3)^T \tag{12}$$

is introduced. The resulting system of first order differential equations is

$$\begin{array}{rcl} x_1 & = & q_1 \\ x_2 & = & q_2 \\ x_3 & = & q_3 \\ x_4 & = & \dot q_1 \\ x_5 & = & \dot q_2 \\ x_6 & = & \dot q_3 \end{array} \Rightarrow \begin{array}{l} \begin{pmatrix} \dot x_1 \\ \dot x_2 \\ \dot x_3 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = x^2 \\ \begin{pmatrix} \dot x_4 \\ \dot x_5 \\ \dot x_6 \end{pmatrix} = M^{-1}(x^1)\,\left(D\,u + \chi^d(x^1,x^2) + \chi^g(x^1)\right). \end{array} \tag{13}$$

For convenience, the functionals J_2 and J_3, resp., are transformed into a Mayer type functional by defining an additional state variable x_7

$$\dot x_7 = \begin{cases} \sum_{i=1}^3 (u_i(t))^2, & J = J_2, \\ \sum_{i=1}^3 (x_{i+3}(t)\,u_i(t))^2, & J = J_3, \end{cases}, \; x_7(0) = 0, \;\Rightarrow\; J[u] = x_7(t_f) \;\to \min! \tag{14}$$

3.1 Necessary Conditions of Optimality

Necessary conditions of optimality are obtained via the minimum principle, cf., e. g., [3]. With the adjoint variables

$$\lambda = (\lambda^1, \lambda^2)^T, \quad \lambda^1 := (\lambda_1, \lambda_2, \lambda_3)^T, \quad \lambda^2 := (\lambda_4, \lambda_5, \lambda_6)^T \tag{15}$$

the Hamiltonian function of the unconstrained problem is

$$\begin{aligned} H^{\text{free}}(x,u,\lambda) \;:=\; & (\lambda^1)^T x^2 + (\lambda^2)^T M^{-1}(x^1)\left(Du + \chi^d(x^1,x^2) + \chi^g(x^1)\right) \\ & + \begin{cases} 0, & J = J_1 \\ \lambda_7 \sum_{i=1}^3 (u_i(t))^2, & J = J_2 \\ \lambda_7 \sum_{i=1}^3 (x_{i+3}(t) u_i(t))^2, & J = J_3 \end{cases}. \end{aligned} \tag{16}$$

To have a uniform treatment of active upper or lower state constraints we introduce the new state constraints

$$S_i \;:=\; q_i^2(t) - q_{i,\max}^2 \;\le\; 0, \quad S_{3+i} \;:=\; \dot q_i^2(t) - \dot q_{i,\max}^2 \;\le\; 0, \quad i = 1,2,3. \tag{17}$$

With the abbreviations for the total time derivatives of S

$$S_i^{(k)} := \frac{\mathrm{d}^k}{\mathrm{d}t^k} S_i, \quad i = 1,\ldots,6, \quad k = 0,1,2,\ldots \tag{18}$$

we find

$$\frac{\partial}{\partial u_i} S_i^{(1)} = 0, \quad \frac{\partial}{\partial u_i} S_i^{(2)} \neq 0, \quad \frac{\partial}{\partial u_i} S_{3+i}^{(1)} \neq 0, \quad i = 1,2,3. \tag{19}$$

The functions S_i are second and the S_{3+i} are first order state constraints. Thus the Hamiltonian becomes (cf. Bryson, Denham, Dreyfus [2])

$$H(x,u,\lambda,\eta) \;=\; H^{\text{free}} + \sum_{i=1}^{3} \eta_i S_i^{(2)} + \sum_{i=1}^{3} \eta_{i+3} S_{3+i}^{(1)}, \tag{20}$$

where $\eta = \eta(t)$ is a multiplier. The necessary conditions from the minimum principle yield for the state and adjoint variables among others (cf. [20])

$$\dot{x}_i \;=\; \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i \;=\; -\frac{\partial H}{\partial x_i}, \qquad \eta_i \geq 0, \quad \eta_i\, S_i = 0, \quad i = 1, \ldots, 6. \tag{21}$$

The optimal control is determined by the minimum principle

$$u^*(t) \;=\; \arg\min_{v \in \mathcal{U}} H(x(t), v, \lambda(t), \eta(t)) \tag{22}$$

where $\mathcal{U}$ denotes the control space. If $J = J_2$ or J_3 then $\lambda_7 \equiv 1$ in $[0, t_f]$ because

$$\dot{\lambda}_7(t) = 0, \quad \lambda_7(t_f) = \frac{\partial J}{\partial x_7(t_f)}, \tag{23}$$

and $S_j^{(k)}$, $k = 0, 1, 2$, and $S_{3+j}^{(l)}$, $l = 0, 1$, $j = 1, 2, 3$, and the right hand sides of the first order differential equations (13), (14) do not depend on x_7.
Furthermore, it can be easily shown (for all three objectives) that $S_i(t) = 0$ and $S_{3+i}(t) \;=\; 0$, $i = 1, 2$, or 3, cannot occur at the same time (cf. [20], [22]).
In the time optimal motion ($J = J_1$) the controls appear linearly in H. Thus H is not regular. If no state constraint is active the i-th optimal control of bang-bang type is determined by the sign of the switching function W_i

$$W_i(t) := (\lambda^2)^T \left[M^{-1}(x^1)\right]_{i\text{–th column}}, \quad u_i(t) = \begin{cases} +u_{i,\max}, & W_i < 0, \\ -u_{i,\max}, & W_i > 0. \end{cases} \tag{24}$$

The case of a *singular* control, i. e., $W_i \equiv 0$ in a whole subinterval, did not occur in the point-to-point trajectories considered here, but might be possible, too.
The minimum time t_f is determined by

$$H|_{t=t_f} \;=\; -\frac{\partial J_1}{\partial t_f} \;=\; -1. \tag{25}$$

As $\mathrm{d}H/\mathrm{d}t = 0$, $t \in [0, t_f]$, it follows that $H \equiv \text{const} = -1$ along the time optimal trajectory.
If the minimum power consumption ($J = J_2$) or the minimum energy criterion ($J = J_3$) are chosen the unbounded optimal control is determined by

$$\frac{\partial}{\partial u_i} H \;=\; 0, \quad i = 1, 2, 3, \tag{26}$$

if $S_i(t) \neq 0$ and $S_{3+i}(t) \neq 0$.
If $J = J_3$ it can be shown that if $\dot{q}_i(0) = 0$ ($i = 1, 2$, or 3) then there exists $\epsilon > 0$ such that $|u_i(t)| = u_{i,\max}$, for $t \in [0, \epsilon]$, and in the same way at t_f (cf. [20], [22]).
If a state constraint is active, e. g., $S_i = 0$ or $S_{3+i} = 0$ then u_i is determined from $S_i^{(2)} = 0$ or $S_{3+i}^{(1)} = 0$, resp., for any objective.
All in all, the necessary conditions can be stated as a well-defined multi-point boundary value problem if the optimal *switching structure* of state and control constraints is known. For more details we refer the interested reader to [20].

3.2 Multiple Shooting Method

The multiple shooting method has shown to be an effective tool in solving highly nonlinear multi-point boundary value problems. The method is described, e. g., by Bulirsch [4] and Stoer, Bulirsch [24]. Its application to a complicated state constrained optimal control problem is described by Bulirsch, Montrone, and Pesch [5]. Here, we used the code BNDSCO due to Oberle [18].
The main drawbacks when applying the multiple shooting method in the numerical solution of optimal control problems are **1.** the derivation of the necessary conditions (e. g., the adjoint differential equations), **2.** the estimation of the optimal switching structure, and **3.** the estimation of an appropriate initial guess of the unknown state and adjoint variables $x(t)$, $\lambda(t)$, $\eta(t)$ in order to start the iteration process. The great advantage of the multiple shooting method is the verification of the optimality conditions resulting in a highly accurate solution.
To overcome the first drawback a good knowledge of optimal control theory is required. Proper estimates of the switching structure and of the adjoint variables might then be provided by the use of a homotopy or continuation technique. This can be a very laborious task (cf. [5] for an example) that is especially cumbersome in our problem as none of the adjoint variables is given either at 0 or t_f. In this paper we will demonstrate how the drawbacks 2 and 3 can be overcome when a direct collocation method is used in a pre-computation to estimate the optimal switching structure, state and adjoint variables. In the derivation of the necessary conditions we used the symbolic computation method MAPLE due to Char et al. [6].

3.3 Direct Collocation Method

The basis of the direct collocation approach is a finite dimensional approximation of control *and* state variables, i. e., a discretization. Here, we choose a continuous, piecewise linear control approximation and a continuously differentiable, piecewise cubic state approximation, cf. Hargraves, Paris [11] and [25], [26], [28]. The differential equations, the state and control constraints are only pointwise fulfilled in this approach. The discretization results in a nonlinear optimization problem subject to nonlinear constraints. Convergence properties of the method and details of an efficient implementation are discussed in [26], [27]. Here, we used the code DIRCOL [27] where the resulting nonlinear programming problems are solved by the Sequential Quadratic Programming method NPSOL due to Gill, Murray, Saunders, and Wright [9]. The direct collocation method has a large domain of convergence and is easy to handle as the user has not to be concerned with adjoint variables or necessary conditions of optimality.

3.4 Combination of the Direct and the Indirect Method

Following [27] and [28] both methods are combined as follows: The direct collocation method is applied with a poor initial guess of the solution $x(t)$, $u(t)$, i. e., with an initial trajectory that interpolates the given values at initial and final time linearly. The obtained (suboptimal) solution provides reliable estimates of

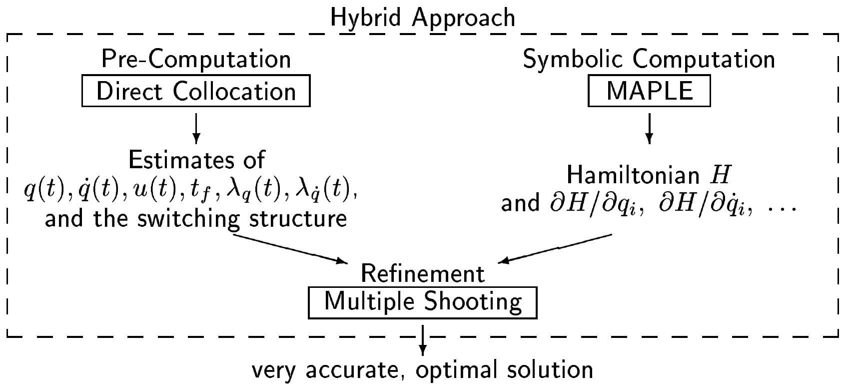

Figure 2: Combination of direct, indirect and symbolic methods in robot trajectory optimization.

the state and adjoint variables and of the switching structure of state and control constraints (cf. [26], [27]). With this guess the multiple shooting method is applied to the multi-point boundary value problem resulting from the necessary conditions of optimality. For the derivation of the necessary conditions we used the symbolic computation method MAPLE [6]: The equations of motion are explicitly given in the appendix of [19] in the form of Eq. (1). First the explicit inverse of the mass matrix $M^{-1}(q)$ is computed. Then the partial derivatives of each component of the vector function $M^{-1}(Du + \chi^d + \chi^g)$ with respect to q_i, $\dot{q}_i$, $i = 1, 2, 3$, are derived. As output of the MAPLE program we obtain a FORTRAN code for M^{-1}, the adjoint differential equations, the Hamiltonian function and the formulae for the boundary controls from $S_i^{(2)} = 0$ or $S_{3+i}^{(1)} = 0$ in the state constrained case. The resulting FORTRAN codes for the inverse mass matrix are about 630 lines and for the adjoint differential equations are about 3350 lines long although in each step of the derivation several optimization strategies are applied in MAPLE to simplify the resulting formulae. For more than three degrees of freedom it might be more efficient to use automatic differentiation [10] and to make even more use of the special structure of the robotic dynamics in order to keep the number of the resulting arithmetic operations as small as possible.

4 Numerical Results

All computations have been performed on a SUN Sparc station 2 with 40 MHz. A special quarter rotation around the base of the robot is investigated with a load of 0 kg

$$q(0) = \begin{pmatrix} 0.00 \\ -1.50 \\ 0.00 \end{pmatrix}, \quad q(t_f) = \begin{pmatrix} 1.00 \\ -1.95 \\ 1.00 \end{pmatrix}, \quad \dot{q}(0) = 0, \quad \dot{q}(t_f) = 0. \tag{27}$$

The direct collocation method is at first applied to the state constrained minimum time problem with $\tilde{q}_i(t) = q(0) + (t/t_f)(q(t_f) - q(0))$, $\dot{\tilde{q}}_i(t) = 0$, $\tilde{u}_i(t) = 0$,

NGRID	NY	NLEQ	NITER	NZJAC	CPU-Sec	t_f
7	58	42	162	33.3%	75	$\underline{0}.56901402$
13	118	84	6	17.1%	43	$\underline{0.49}689417$
21	198	140	9	10.4%	145	$\underline{0.495}74811$
47	458	322	6	4.5%	960	$\underline{0.495}21483$
81	798	560	6	2.6%	5248	$\underline{0.495}23283$
NDEQ	NKNOT	NS	NITER	–	CPU-Sec	t_f
19	15	8	14	–	393	0.49518904

Table 1: Convergence history of the first time optimal motion.

$i = 1, 2, 3$, $t \in [0, t_f]$ and $\tilde{t}_f = 1$[s] as initial estimates of the unknown solution. A first solution is obtained for 7 equidistant grid points. A refinement of the discretization yields a sequence of nonlinear programming problems with increasing dimensions ending up in an 81 grid point solution. The convergence history is shown in Table 1 where NGRID is the number of grid points of the direct collocation method DIRCOL, NY is the number of degrees of freedom of the nonlinear program, NLEQ is the number of nonlinear equality constraints of the nonlinear program, NITER is the number of iterations of the SQP-method NPSOL [9], NZJAC is the percentage of non zero Jacobian elements in the nonlinear program, CPU-Sec is the computing time in seconds, NDEQ is the number of differential equations of the multi-point boundary value problem of the necessary conditions, NKNOT is the number of multiple shooting nodes used in BNDSCO, NS is the number of switching points, NITER is the number of Newton iterations in BNDSCO.

Remark 1. The inequality constraints on state and control variables are treated as box constraints in the nonlinear program. Therefore no nonlinear inequality but only equality constraints appear in the discretized problem.

Remark 2. To compare the time optimal with the minimum energy solution a seventh state variable x_7 has been used in the computations. Therefore the computing time for the time optimal motion will in fact be less than the reported times if x_7 is not computed in the optimization.

Remark 3. It is a remarkable fact the the number of SQP-iterations is not increasing with the number of degrees of freedom in the nonlinear program. The increase in computing time is due to the fact that the sparsity patterns in the gradients are not yet used in the linear algebra of the quadratic programming subproblems. Much efficiency can still be gained if an appropriate sparse linear algebra is used (cf. Betts, Huffman [1] and Gill [8]).

From the solution for 81 grid points the switching structure is guessed, i. e., number and type of the switching points. With this switching structure and the state and the estimated adjoint variables from the direct collocation method the multiple shooting method is successfully applied to solve the multi-point boundary value problem of optimality conditions. The solution of the direct collocation

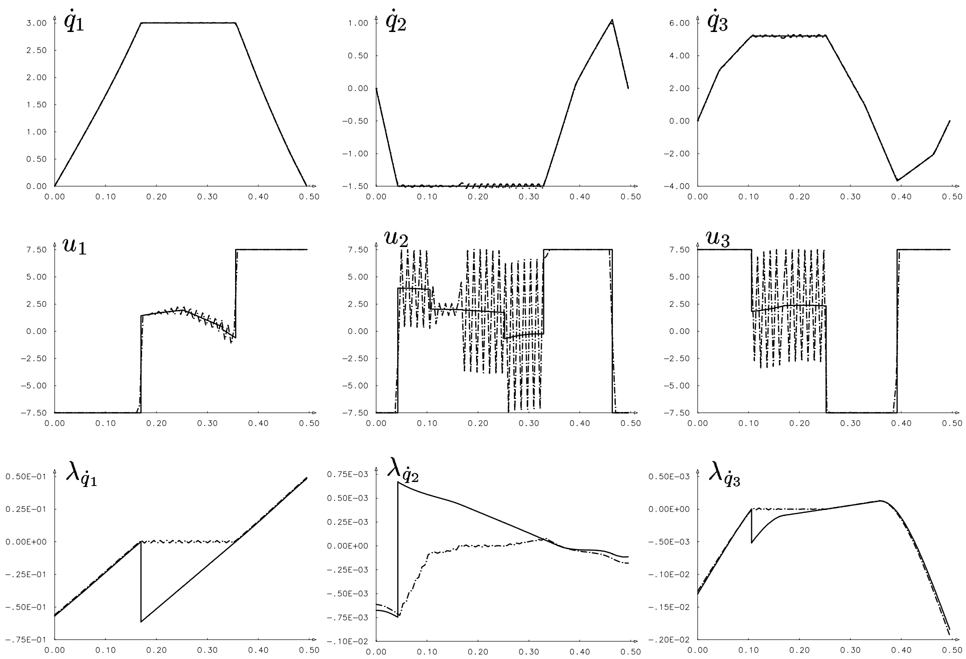

Figure 3: Solution curves for the state constrained time optimal motion of the direct collocation method (–·–·–) compared with the multiple shooting method (———). In the curves of the angles $q_i(t)$ there are no visible differences between the two solutions. They are shown in Figure 5.

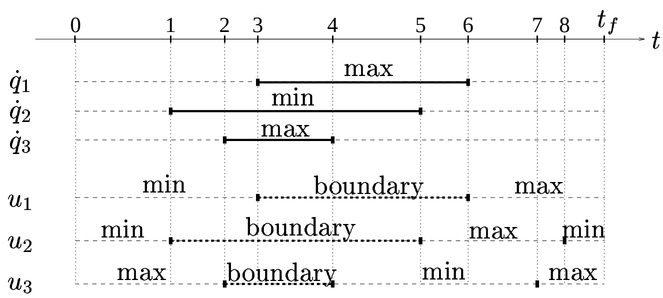

Figure 4: Switching structure of the state constrained time optimal motion.

method is shown with the solution of the multiple shooting method in Figure 3, the initial guess and the finally obtained switching points are listed in Table 2, and the qualitative behaviour of the switching structure is shown in Figure 4. The three dimensional motion of the robot is shown in Figure 6.

Remark 4. The oscillating behaviour of the discretized controls results from the oscillating behaviour of the only pointwise fulfilled state constraints on $\dot{q}_i$. This is

sw. point	1	2	3	4	5	6	7	8
estimated	.0495	.1114	.1733	.2538	.330	.359	.390	.464
final	.04248	.10585	.16965	.25243	.3289	.3556	.3919	.4638

Table 2: Estimated and final switching points of the state constrained time optimal motion.

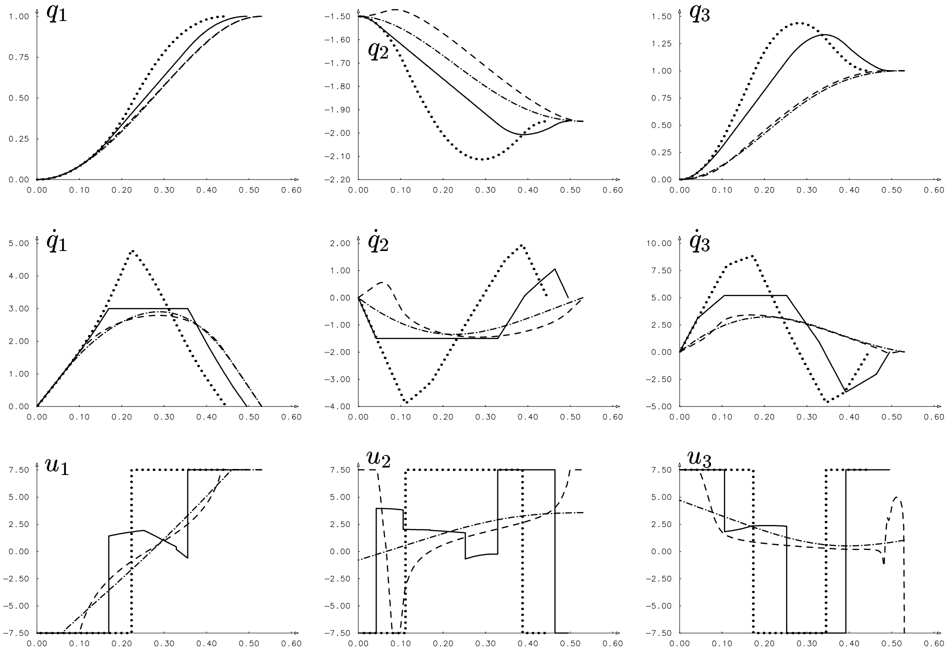

Figure 5: Solution curves of state and control variables for the state constrained time optimal motion (———), the minimum power consumption (– – –), the minimum energy motion (–·–·–), and the unconstrained time optimal motion (·····).

trajectory\criterion	t_f	$\int_0^{t_f} \sum_i u_i^2(t)\mathrm{d}t$	$\int_0^{t_f} \sum_i (q_i(t)u_i(t))^2\,\mathrm{d}t$
state cons. min. time	$\underline{0.495}18904^\star$	51.351466	248.56509
uncons. min. time	$\underline{0.4455}1780^\star$	75.179701	906.30989
min. energy	0.53000000	$\underline{20.4042}47^\star$	43.089470
min. power consump.$^+$	0.53000000	28.057499	$35.911668^\star$

Table 3: Comparison of results of the multiple shooting method for different objectives (correct figures of the direct collocation method underlined, * denotes the optimum value, $^+$ solution only computed by direct collocation).

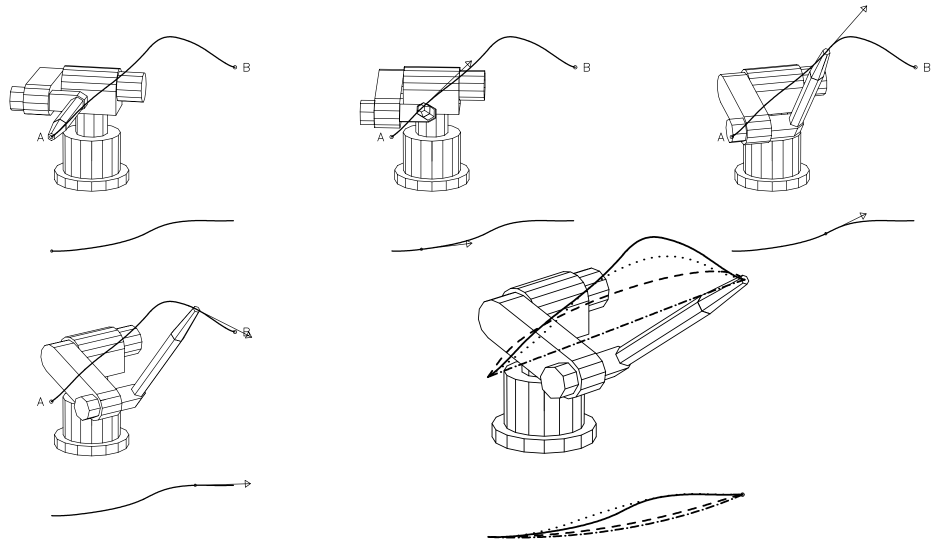

Figure 6: Five snapshots of the state constrained time optimal motion from A ($t = 0$) to B ($t = t_f$) at $t = i\,t_f/4,\ i = 0, 1, 2, 3, 4$, and a comparison of the four trajectories.

a common behaviour of direct collocation or direct shooting methods. Also, when entering the constraint on $\dot{q}_i$ the control u_i is not continuous and the angular velocity $\dot{q}_i$ is only continuous but not differentiable at the entry (and exit) point of the state constraint. Therefore a better approximation of state and control variables can be obtained either by increasing the number of grid points to a huge number or by taking the *switching structure* into account in a so-called *multi-stage* or *multi-phase* discretization, too ([26], [27]). I.e., the switching points are introduced as additional variables and the controls are allowed to be changed discontinuous and the states to be changed not differentiable at the switching points. This adopted discretization results in a more accurate solution of the direct collocation method for the controls and the objective value with even less grid points (cf. [26] for an example and [27] for more details). The non adopted standard discretization is shown in Figure 3 to demonstrate the oscillating behaviour on state constraint subarcs.

Remark 5. The adjoint variables and their estimates seem to differ significantly on state constrained subarcs (cf. Figure 3). The estimates of adjoint variables obtained by the direct collocation method are related to the adjoint variables from the necessary conditions of Jacobson, Lele, and Speyer [12] where the state constraints themselves are coupled to the free Hamiltonian with a multiplier function $\bar{\eta}$

$$H(x, u, \lambda, \bar{\eta}) \;=\; H^{\text{free}} + \sum_{i=1}^{3} \bar{\eta}_i S_i + \sum_{i=1}^{3} \bar{\eta}_{i+3} S_{3+i}. \qquad (28)$$

For the formulation of the multi-point boundary value problem the necessary conditions of Bryson, Denham, and Dreyfus (Eq. (20)) have been used. Both sets of adjoint variables differ only along the state constrained subarcs and can be transformed into each other [17], [27].

When the state constrained minimum time motion has been computed, solutions to the minimum power consumption and the minimum energy criterion can be computed to a prescribed final time that is only about 10-15 % slower than the minimum time and the constraints on the angular velocities do not become active. Here, we choose $t_f = 0.530$[s] that is 7 % slower than the minimum time. The solution curves are shown in Figure 5 and compared with the minimum time solution.

To analyze the impact of the constraints on the angular velocities on the time optimal motions, the minimum time solution is computed where the state constraints are not taken into account. The resulting minimum time is 10 % faster than the state constrained minimum time solution. But this solution violates the constraints on $\dot{q}_i(t)$, $i = 1, 2, 3$, and on $q_2(t)$ and is shown in Figure 5. Thus the constraints on the angular velocities play an important role in the time optimal motion.

Remark 6. All solutions shown in Figure 5 have been obtained by the combination of the direct and the indirect method besides the solution for the minimum power consumption criterion. The only visible significant differences between the solutions of both methods are in the state constrained time optimal motion and are shown in Figure 3. The differences in the objective values are listed in Table 3.

5 Conclusions

It has been demonstrated that even a simple manoeuvre can exhibit quite a difficult switching structure in the time optimal motion of an industrial robot. The state constraints on the angular velocities play an important role in the time optimal motion as they often become active. The knowledge of the fastest possible motion provides reliable bounds for fast minimum energy motions. Hereby, the stress on the links of the robot can be significantly decreased if an increase in time of about ten percent is accepted. Thus lifetime and reliability of the robot will increase.

The second link of the Manutec r3 robot is the weakest. This is indicated also by several other time optimal movements investigated by the authors where the constraints on $\dot{q}_2(t)$ become active during most parts of the motions. Thus a better design of robots might be possible if the investigation of optimal trajectories is included in the development phase.

The combination of direct and indirect methods, namely direct collocation and multiple shooting, is an efficient hybrid approach for solving highly complex, nonlinear, real life optimal control problems that amalgamates the benefits of both methods.

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