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Optimal Control of Investment, Level of Employment and Stockkeeping[1]

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Extended Abstract

Numerical simulations of microeconomic processes usually suffer from poor models and therefore often can not exhibit real phenomenons. We present a new, very complex model of an enterprise – deduced from the model of Lesourne at al. (1978) and Feichtinger at al. (1986), see Koslik et al. (1993) – including own capital distributed among the enterprise (part X) and alternative capital assets (part X_m), outside capital Y, level of employment L, a stock S and a submodel with production and selling. The interest rate for outside capital ρ_k and the interest rate for alternative own capital assets ρ_m are taken from the real values of the last ten years (May 1983 to May 1993)[2]. For the inflation i we take $i := \rho_m - 0.05$. For the own capital X, the management of the enterprise has to pay a risk premium $\rho_{r,1}\,X$ or $\rho_{r,2}\,X$ dependend on the risk of the business.

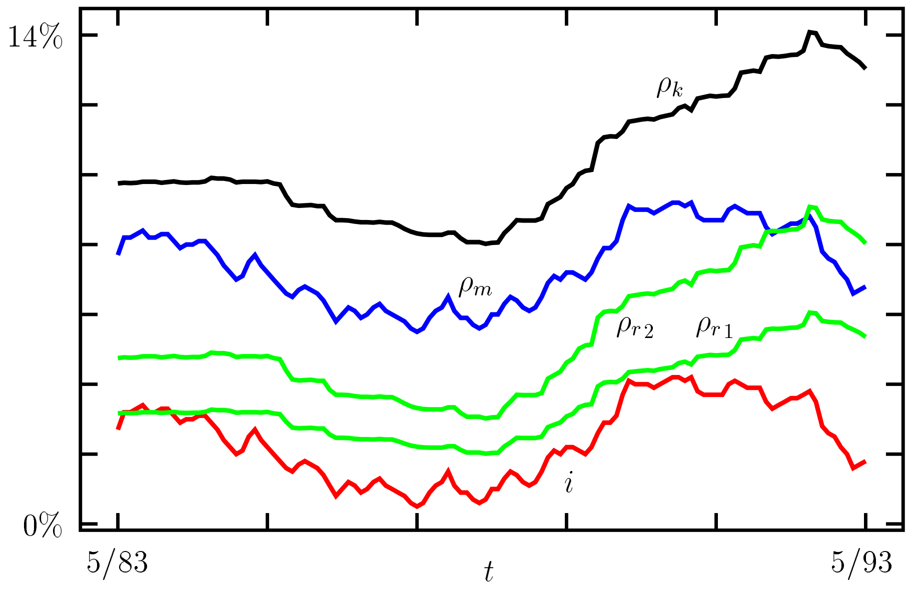

Fig. 1: Interest rate for outside capital ρ_k, interest rate for alternative own capital assets ρ_m, risk premium rates for the own capital in the enterprise X ρ_{r1} and ρ_{r2} and inflation rate i.

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[2]The authors would like to thank P. Sollinger, HYPO-Bank Zentrale München, for providing us with the interest rates ρ_k and ρ_m for the last ten years.

The whole enterprise can be described by the following differential equations

$$\begin{aligned}
\dot{S} &= S_c \ , \\
\dot{L} &= L_c \ , \\
\dot{Y} &= Y_c \ , \\
\dot{X} &= +I + (1-\tau)\cdot(P - \rho_r \cdot X) \ , \\
\dot{X}_m &= -I + (1-\tau)\cdot X_m \cdot \rho_m \ , \\
\dot{X}_r &= (1-\tau) \cdot \rho_r \cdot X \ , \\
\dot{d} &= -d \cdot \ln(1+i) \ ,
\end{aligned}$$

with

$$F = \alpha \cdot (X+Y)^{\alpha_K} \cdot L^{\alpha_L} \ ,$$

and

$$P = \frac{1}{d} \cdot (p \cdot (F - S_c) - \sigma \cdot S - \omega \cdot L) - \rho_K \cdot Y - \delta \cdot (X+Y) \ .$$

Here the dot denotes the derivative w. r. t. the time $t \in [0, t_f]$. The horizon t_f is fixed at 10 years. $S_c \in [S_{c\,\min}, S_{c\,\max}]$ is the control for the stock S, $L_c \in [L_{c\,\min}, L_{c\,\max}]$ is the control for the employment level L, $Y_c \in [Y_{c\,\min}, Y_{c\,\max}]$ is the control for the outside capital Y and the investment $I \in [I_{\min}, I_{\max}]$ is the control for the own capital distribution among the enterprise X and alternative capital assets X_M. τ is the tax rate, F is the output, p the selling price, σ are the storage charges, ω are the labor costs, δ is the rate of depreciation, P is the profit and d is the reciprocal total inflation since $t = 0$. The simple bounds for the state variables must not be violated at any time $t \in [0, 10] : 50 \le S \le 200$, $0 \le L$, $0 \le Y$, $0 \le X$ and $0 \le X_m$. In addition the important borrowing limit $Y \le 0.8 \cdot X$ and the minimum production limit $F \ge S_c$ must be obeyed. The management of the enterprise tries to maximize the total profit at the end of the planing period (May 1993) with the help of the control functions $S_c(t)$, $L_c(t)$, $Y_c(t)$ and $I(t)$. Therefore the goal in the optimal control problem can be formulated as

$$X(t_f) + X_m(t_f) + (1-\tau)\cdot p \cdot \frac{S(t_f)}{d(t_f)} \quad \longrightarrow \quad \textbf{max !} \ .$$

The optimal control problem has been solved with the help of the new collocation method DIRCOL, see von Stryk 1992. The major advantages of this direct method are:

- Non differentiable model functions, e.g. ρ_k and ρ_m, can be handled, since no explicit numerical integration of the differential equations is done.
- The large domain of convergence enables the computation of the optimal solution even with a poor initial guess.
- Although the calculation of the adjoint differential equations is not necessary, the direct collocation method DIRCOL yields accurate estimates for the adjoint variables. These estimates facilitate the use of an highly accurate indirect optimization method, e.g. the multiple shooting method.

The optimal management of the enterprise under consideration for the last ten years can be seen in Figs. 2 – 9, where the dots are placed at the position of the collocation nodes.

Figs. 2 – 8 show the optimal solution for $\rho_r = \rho_{r,1}$ and Fig. 9 enables the comparison for $\rho_r = \rho_{r,2}$. The optimization of the enterprise requires about one and a half year in both cases. After this initial phase the influence of the economic cycles on the optimal capital partition can be obtained in Fig. 2 and Fig. 9 and the optimal controls I and Y_c for the capital flow are depicted in Fig. 3 and Fig. 4. Note that the optimal linear controls I and Y_c are controlled singular most the time.

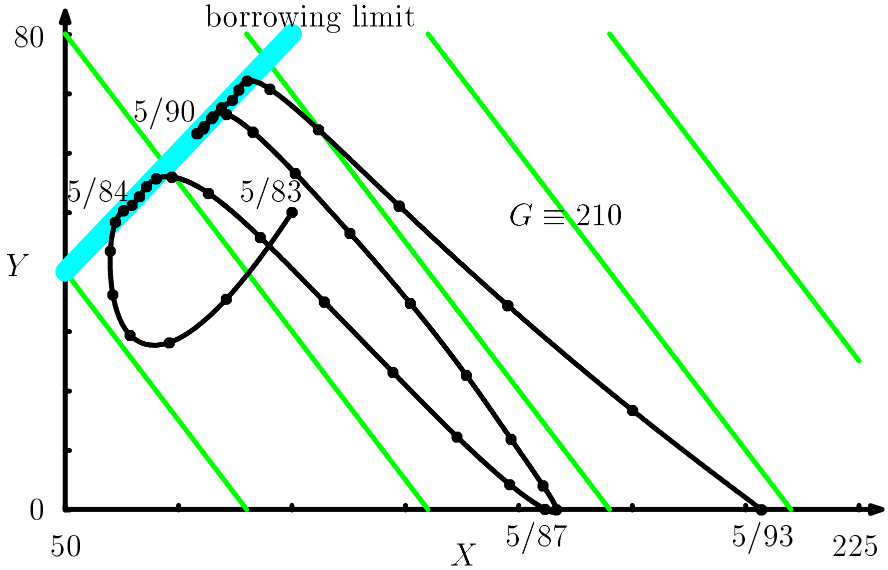

Fig. 2: Risk premium $\rho_r = \rho_{r1}$. Outside capital Y versus own capital in the enterprise X (black curve), borrowing limit (thick gray line) and lines of constant joint capital $G := X + Y$ (thin gray lines).

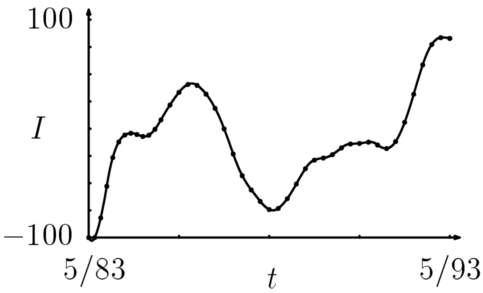

Fig. 3: Control I for the own capital partition.

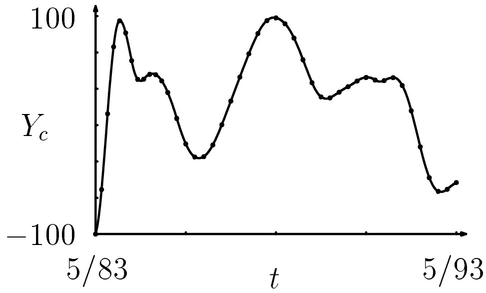

Fig. 4: Control Y_c for the outside capital.

The optimal level of employment and the optimal stock are shown in Fig. 5 and Fig. 6 .

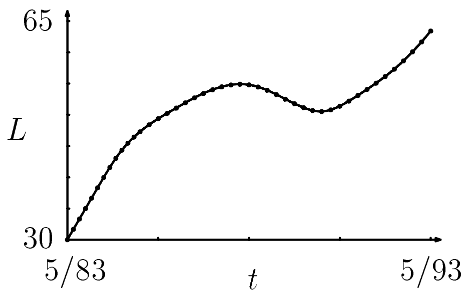

Fig. 5: Level of employment L.

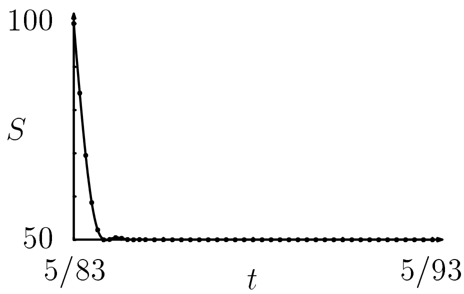

Fig. 6: Stock S.

The part of the own capital invested in alternative capital assets, e.g. fixed-interest securities, is shown in Fig. 7. The reciprocal total inflation d, see Fig. 8, enables the computation of the actual prices and costs at any time for a varying inflation.

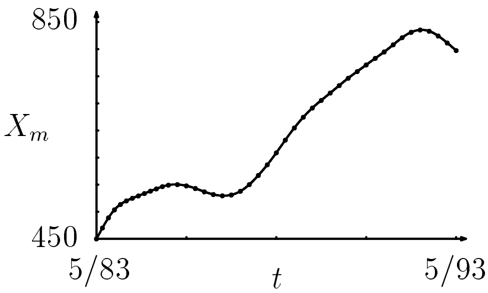

Fig. 7: Alternative own capital assets X_m.

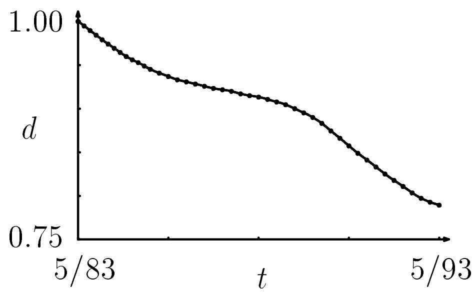

Fig. 8: Reciprocal total inflation d.

For $\rho_r = \rho_{r2}$ the borrowing limit $Y \leq 0.8 \cdot X$ once reached stays active for all t, see Fig. 9.

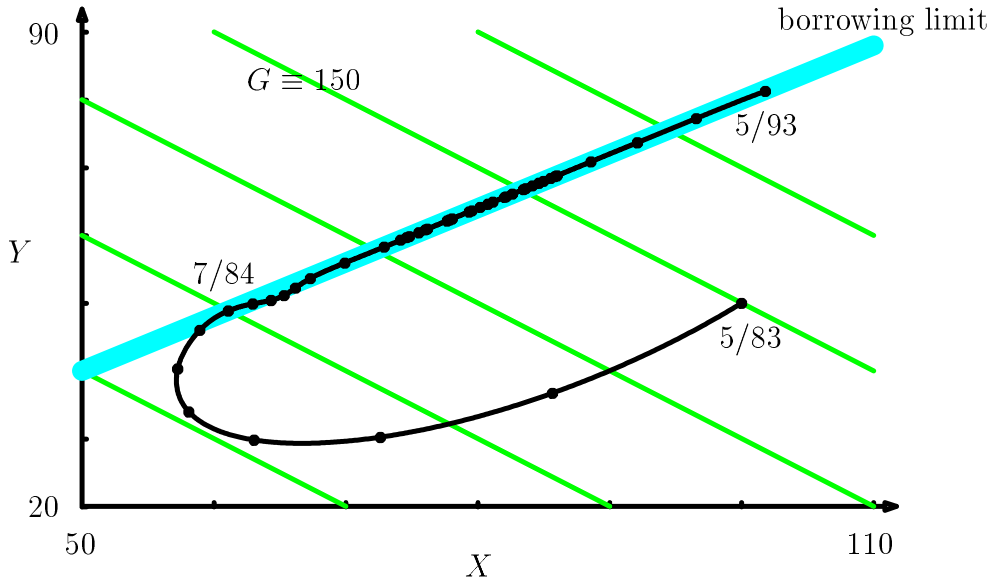

Fig. 9: Risk premium $\rho_r = \rho_{r2}$. Outside capital Y versus own capital in the enterprise X (black curve), outside capital limit (thick gray line) and lines of constant joint capital $G := X + Y$ (thin gray lines).

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